Feynman Rules

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Introduction

Most textbooks don't have satisfactory descriptions of deriving the Feynman rules in my opinion. The first camp of authors is Weinberg, who derives things rigorously, but the rules break down since his rules forces one to consider a larger number of diagrams. The other camp is Schwartz, Srednicki, and Peskin and Schroeder, who derive propagators rigorously, but vertex factors are only derived by considering external legs on the mass shell, which is clearly not sufficient for a general diagram. However, this second camp's rules are much cleaner and make computing diagrams simpler. The point of this note is to make contact between these two camps.

To say a bit more on Weinberg's philosophy, take the self-interaction of a real scalar field, for example. The interaction Lagrangian is

$$\mathscr{L}_{\text{int}} = -g\phi\partial_\mu\phi\partial^\mu\phi. \tag{1}$$

In Weinberg's view, the vertex factor is always simply $-ig$, but one has three different propagators, corresponding to pairings of ϕ with ϕ, $\partial_\mu\phi$ with ϕ, and $\partial_\mu\phi$ with $\partial_\nu\phi$. Then, for tree-level $2 \to 2$ scattering to second order in g, while there are only two vertices, one must consider three distinct types of diagrams corresponding to the different propagators that could connect the vertices. This clearly has the issue of scaling for more general diagrams.

However, Weinberg's textbook actually has some tension with itself. In the path integral section, he considers propagators between vertices (in position space) as the inverse of the coupling matrix of the fields in the free Lagrangian (i.e. the quadratic term). Of course, there is only one such term, and therefore there should only be one such propagator. So where do the extra factors associated with derivatives go? They must be associated with each vertex. However, he doesn't give a clear prescription on how to make this adjustment.

The impetus of this note was really to understand scalar QED, which has two issues. First, the derivative couplings which introduce the increased number of propagators. The second is that the interaction term with field derivatives is actually two terms:

$$\mathscr{L}_{\text{int}}^{\text{Scalar QED}} = a_\mu \left(ie((\partial^\mu\phi^\dagger)\phi - \phi^\dagger\partial^\mu\phi) \right) \tag{2}$$

which arises due to the field being complex rather than real. So, to go with Weinberg's approach, we are stuck with four propagators and two vertices. Contrast this with Schwartz, Srednicki, and Peskin and Schroeder, who all give Feynman rules for scalar QED with a single propagator and a single vertex factor, albeit derived in a non-general way.

Of course, scalar QED is not a theory actually realized by nature, so is it worth studying? Yes, because nature does realize derivative interactions, but in a more complicated theory, the electroweak theory. The three-point interaction for the charged W bosons with the photon is a derivative interaction. So, a rigorous understanding of how to derive the Feynman rules in theories with complex fields and derivative couplings is crucial to understand our actual world.

To warm up, I will consider a theory of real scalar fields with a derivative coupling and a theory of complex scalars with no derivative couplings before tackling scalar QED. Also, in all of these, I will neglect the $i\epsilon$ terms and only give the classical Lagrangian.

1 Real Scalar Field, Derivative Coupling

Take the action to be

$$\begin{aligned} I[\phi] &= I_0[\phi] + I_1[\phi] \\ I_0[\phi] &= \int d^4x \left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \right] \\ I_1[\phi] &= \int d^4x \left[-g\phi\partial_\mu\phi\partial^\mu\phi \right] \end{aligned} \tag{3}$$

To find the propagator, recall that we want to put the free part of the action into the form

$$I_0[\phi] = -\frac{1}{2}\int d^4x d^4y \mathscr{D}_{x,y}\phi(x)\phi(y). \tag{4}$$

Weinberg has thankfully already done this in chapter 9.4, so to quote him:

$$\begin{aligned} \mathscr{D}_{x,y} &= \frac{1}{(2\pi)^4}\int d^4p e^{ip\cdot(x-y)}(p^2+m^2-i\epsilon) \\ \implies \mathscr{D}^{-1}_{x,y} = \Delta(x,y) &= \frac{1}{(2\pi)^4}\int d^4p e^{ip\cdot(x-y)}\frac{1}{p^2+m^2-i\epsilon} \end{aligned} \tag{5}$$

Next, we need to consider how to find the vertex factor. This is nontrivial, so let's spend some time on it.

In any theory, a vertex within a larger diagram will have legs that are external, internal, or a combination of both. Each leg can either be on-shell or off-shell. The most general vertex comes from when all legs are off-shell, since any leg can be made on shell via the LSZ reduction formula: Strip away the propagator associated with that field and replace it with a field coefficient $u_l(\mathbf{p})$. An arbitrary vertex within a larger diagram can then be grouped into one of the following categories:

1. All legs are internal and go to different vertices, or some legs are external, but we know they will be replaced with field coefficients via the LSZ reduction formula when calculating the matrix element we care about *or* they are simply external fields and not external derivatives of fields.

2. Same as item 1, but some legs may meet at the same vertex, forming a loop.

3. Some legs are internal and may extend the different vertices or the same vertex, the other legs are external, and for at least one external leg, we are not allowed to take it on the mass-shell when calculating the matrix element we care about, *and* it is a derivative of a field. An example of this is if that leg represents a some fixed external field.

What this note will calculate is the vertex factor associated with item 1. Item 2 can be calculated similarly to item 1, but will yield slightly different combinatorics. Item 3 is very hard to generalize, as it depends on the external field, so we will not consider it, but can be considered on a case-by-case basis.

Since we only care about item 1, it is sufficient to look at the vacuum-to-vacuum correlator. This is because all legs are internal and off-shell. The vertex factors found from this amplitude will generalize to diagrams with external legs that can be put on the mass-shell. The sum of all vacuum-to-vacuum diagrams is

$$\begin{aligned}\langle \text{VAC, out}|\text{VAC, in}\rangle &= \int \left(\prod_x d\phi(x)\right) e^{iI[\phi]} \\ &= \int \left(\prod_x d\phi(x)\right) e^{iI_0[\phi]} \sum_{N=0}^{\infty} \frac{i^N}{N!} \left(I_1[\phi]\right)^N\end{aligned} \tag{6}$$

We wish to find the vertex factors in a single diagram, so it is sufficient to consider only a specific term N in the sum. The next thing we want to do is massage the interaction action into a form that has no derivatives hitting the fields. We can do this by integrating by parts:

$$\begin{aligned}I_1[\phi] &= -g \int d^4x d^4y d^4z \delta^4(x-y)\delta^4(x-z)\phi(x)\frac{\partial}{\partial y_\mu}\phi(y)\frac{\partial}{\partial z^\mu}\phi(z) \\ &= -g \int d^4x d^4y d^4z \frac{\partial}{\partial y_\mu}\delta^4(x-y)\frac{\partial}{\partial z^\mu}\delta^4(x-z)\phi(x)\phi(y)\phi(z)\end{aligned} \tag{7}$$

Let us designate the variables x, y, z in the i-th factor of I_1 as x_i, y_i, z_i (with similar labels for the fields too). We therefore have for the N-th term in the correlator

$$\begin{aligned}\langle \text{VAC, out}|\text{VAC, in}\rangle_N = \frac{(-ig)^N}{N!} &\prod_{i=1}^{N} \left(\int d^4x_i d^4y_i d^4z_i \frac{\partial}{\partial y_{i,\mu}}\delta^4(x_i-y_i)\frac{\partial}{\partial z_i^\mu}\delta^4(x_i-z_i)\right) \\ &\times \int \left(\prod_x d\phi(x)\right)\left(\prod_{i=1}^{N}\phi(x_i)\phi(y_i)\phi(z_i)\right) e^{iI_0[\phi]}\end{aligned} \tag{8}$$

With the derivatives safely moved outside of the path integral, we can now use the integral expressions in the Appendix of Chapter 9. The amplitude is thus

$$\begin{aligned}\langle \text{VAC, out}|\text{VAC, in}\rangle_N \propto \frac{(-ig)^N}{N!} &\prod_{i=1}^{N} \left(\int d^4x_i d^4y_i d^4z_i \frac{\partial}{\partial y_{i,\mu}}\delta^4(x_i-y_i)\frac{\partial}{\partial z_i^\mu}\delta^4(x_i-z_i)\right) \\ &\times \sum_{\text{pairings}} \prod_{\text{pairs}} \left[-i\mathscr{D}^{-1}\right]\end{aligned} \tag{9}$$

where the sum over pairings corresponds to all the different ways of pairing the fields in the product

$$\prod_{i=1}^{N} \phi(x_i)\phi(y_i)\phi(z_i). \tag{10}$$

We also include the proportionality symbol because there is a functional determinant that doesn't affect the vertex factors.

There are two things to point out. First, since we haven't integrated over the variables y_i and z_i (i.e. we haven't recovered our original expressions before the delta functions were added), the product of propagators represents $3N/2$ disconnected propagators. Of course, the N-th term of the vacuum-to-vacuum amplitude is zero if N is odd. Second, this sum over pairings is Wick's theorem, so each term in the sum corresponds to a specific Feynman diagram (up to an overall permutation of the original x_i vertex labels, which the $1/N!$ takes care of). So, to get the vertex factors, we only need to look at a single term. However, it will be useful to consider certain related groupings of terms.

To find the vertex factors, we need to move to momentum space. We can do this by using Equation 9.4.17 in Weinberg, that says the propagator $\Delta_{l_1,l_2}(x,y)$ from point (y,l_2) to (x,l_1) is the Fourier transform:

$$\Delta_{l_1,l_2}(x,y) = \frac{1}{(2\pi)^4}\int d^4p e^{ip\cdot(x-y)}\mathscr{D}^{-1}_{l_1,l_2}(p) \tag{11}$$

Therefore, in each term of the sum over pairings, there are only two portions of the product that depend on any given y_i or z_i, namely the delta function derivative $\frac{\partial}{\partial y_{i,\mu}}\delta^4(x_i - y_i)$ and the exponential of the propagator. We can then do integration by parts to move the derivative from the delta function to the exponential. Since there are two derivatives per $I_1[\phi]$, we do not pick up any extra minus signs since there will be an even number of integrations by parts.

The effect of the integration by parts is to pull down a factor of $\pm ip^\mu$ where the momentum p^μ is associated with the propagator and the $\pm$ is determined by whether the derivative was with respect to a "destination" point x $(+)$ or a "source" point y $(-)$ in Equation 11. After these derivatives act on the exponential, we are just left with a bunch of propagators, a bunch of delta functions, and some constants that can factor out of all the integrals.

Now, what comes next is specific to this specific case of a real scalar field. Every single possible pairing in the sum over pairings is unique. However, once we integrate over the y_i and z_i, there will be pairings that are equivalent except for the factors of momenta. It is easiest to see this with an example. Consider a subset of the possible $3N$ fields:

$$...\phi(x)\phi(y)\phi(z)\phi(w_1)\phi(w_2)\phi(w_3)... \tag{12}$$

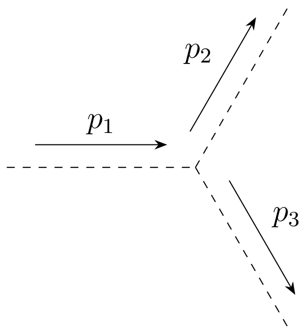
where the x, y, z represent the x_i, y_i, and z_i from above for a specific choice of i, and the w_j can be any possible other coordinates. What this represents is how a single vertex's fields can contract with other fields.

If we restrict ourselves to only look at pairings that don't have self-pairing between the x, y, and z, then there are $3! = 6$ ways of pairing these fields together. However, if we were to integrate over y and z, these six terms would give identical products of propagators, but they may differ by the momenta pulled down from the derivatives. Let $p^\mu_{x,1}$ be the momentum associated with the propagator $\Delta(x, w_1)$, and so forth for other subscripts $(y,1)$, $(z,2)$, etc. The six prefactors of momenta for the six pairings from above are

$$\begin{aligned} &ip_{y,1}\cdot ip_{z,2}, \qquad ip_{y,1}\cdot ip_{z,3} \\ &ip_{y,2}\cdot ip_{z,1}, \qquad ip_{y,2}\cdot ip_{z,3} \\ &ip_{y,3}\cdot ip_{z,1}, \qquad ip_{y,3}\cdot ip_{z,2} \end{aligned} \tag{13}$$

up to an overall sign depending on the direction of momenta flow. When we do the integrals over y and z, we see that the term $ip_{y,1}\cdot ip_{z,2}$ is the same as $ip_{y,2}\cdot ip_{z_1}$, etc. Finally, there is the integral over the x_i. The x_i are only found in the exponentials, so this integral serves to conserve momentum at each vertex, giving a factor of $(2\pi)^4\delta^4(\sum p_i)$.

Everything else about the six terms in the sum over pairings is identical besides these prefactors, so we can factor all the other stuff out. Combining this factor with one factor of $(-ig)$, and also remembering our sign conventions for momenta flow, the vertex factor for this interaction is



$$= -2ig(p_1\cdot p_2 + p_1\cdot p_3 - p_2\cdot p_3)(2\pi)^4\delta^4(p_1 - p_2 - p_3) \tag{14}$$

The important thing here is that the p_i^μ are all off-shell. We can check this answer by considering the momenta on shell for a three-point interaction by calculating

$$\int d^4x \left[-ig \langle 0| a(\mathbf{p}_2) a(\mathbf{p}_3)\phi(x)\partial_\mu\phi(x)\partial^\mu\phi(x) a^\dagger(\mathbf{p}_1)|0\rangle\right] \tag{15}$$

which indeed yields the same answer as above.

2 Complex Scalar Field, Scalar Coupling

The purpose of this section is to understand how to work with a complex scalar field in the path integral approach. Take the action to be

$$\begin{aligned} I[\phi,\phi^\dagger] &= I_0[\phi,\phi^\dagger] + I_1[\phi,\phi^\dagger] \\ I_0[\phi,\phi^\dagger] &= \int d^4x \left[-\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi\right] \\ I_1[\phi,\phi^\dagger] &= \int d^4x \left[-g(\phi^\dagger\phi)^2\right] \end{aligned} \tag{16}$$

The issue with this free action is that there are two degrees of freedom, ϕ and $\phi^\dagger$, and it is not immediately obvious where the quadratic fields are. Indeed, if one were to expand in the real and imaginary parts of ϕ and $\phi^\dagger$, we would recover two copies of the real scalar field action. However, our interaction term is written as a function of ϕ and $\phi^\dagger$, and multiplying out the real and imaginary parts would create a more complicated interaction term.

Instead, let's do a trick. Let's define a new field ϕ_a with the property that

$$\phi_1(x) = \phi(x), \quad \phi_2(x) = \phi^\dagger(x) \tag{17}$$

This has the nice effect that we can write the free action as quadratic in this new field ϕ_a.

$$\begin{aligned} I_0[\phi,\phi^\dagger] &= -\int d^4x d^4y \delta^4(x-y)\left[\frac{\partial}{\partial x^\mu}\phi^\dagger(x)\frac{\partial}{\partial y_\mu}\phi(y) + m^2\phi^\dagger(x)\phi(y)\right] \\ &= -\int d^4x d^4y \phi^\dagger(x)\left[\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y_\mu} + m^2\right)\delta^4(x-y)\right]\phi(y) \\ &= -\frac{1}{2}\int d^4x d^4y \phi_a(x)\left[\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y_\mu} + m^2\right)\delta^4(x-y)\epsilon_{ab}\right]\phi_b(y) \end{aligned} \tag{18}$$

where $\epsilon_{ab} = 1$ for $a \neq b$, and 0 for $a = b$. This has the same form as the first Pauli matrix, but to my knowledge there is no fundamental reason for this, and so I will use this ϵ notation instead of σ_1.

I claim the propagator is

$$\Delta_{ab}(x,y) = \frac{1}{(2\pi)^4}\int d^4p e^{ip\cdot(x-y)}\frac{1}{p^2+m^2-i\epsilon}\epsilon_{ab} \tag{19}$$

Which we can check via:

$$\begin{aligned} &\int \frac{d^4y d^4p}{(2\pi)^4}\left[\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y_\mu} + m^2\right)\delta^4(x-y)\right] e^{ip\cdot(y-z)}\frac{1}{p^2+m^2-i\epsilon}\epsilon_{ab}\epsilon_{bc} \\ &= \int \frac{d^4y d^4p d^4q}{(2\pi)^8}\left(q^2+m^2\right)e^{iq\cdot(x-y)}e^{ip\cdot(y-z)}\frac{1}{p^2+m^2-i\epsilon}\delta_{ac} \\ &= \int \frac{d^4p d^4q}{(2\pi)^4}\left(q^2+m^2\right)e^{iq\cdot x - ip\cdot z}\frac{1}{p^2+m^2-i\epsilon}\delta^4(p-q)\delta_{ac} \\ &= \int \frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-z)}\delta_{ac} \\ &= \delta^4(x-z)\delta_{ac} \end{aligned} \tag{20}$$

where we used $\epsilon_{ab}\epsilon_{bc} = \delta_{ac}$.

The physical significance of the ϵ_{ac} is that it tells us the propagator is zero when connecting two particle or two anti-particle legs, but is non-zero when connecting a particle to an anti-particle leg. This is just the conservation of charge associated with the $U(1)$ symmetry in disguise.

To find the vertex factor, we can use the equations of the Section 1 to construct the contribution to the vacuum-to-vacuum matrix element to N vertices:

$$\begin{aligned}\langle \text{VAC, out}|\text{VAC, in}\rangle_N = \frac{(-ig)^N}{N!}\prod_{i=1}^{N}\left(\int d^4x_i\right)\int\left(\prod_{x,e} d\phi_e(x)\right) \\ \times\left(\prod_{i=1}^{N}\phi_1(x_i)\phi_1(x_i)\phi_2(x_i)\phi_2(x_i)\right)e^{iI_0[\phi]}\end{aligned} \tag{21}$$

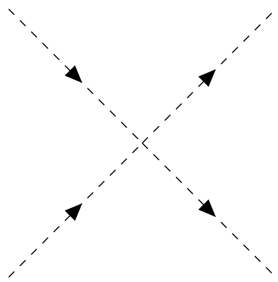
where i labels the interaction vertex of the N vertices.

The path integral evaluates to the sum over all possible pairings of the $4N$ fields in the second line of equation 21 where each pairing gives a propagator $-i\Delta_{ab}(x, y)$. As said before, terms in this sum drop out if a $\phi_1(x)$ field is paired with a $\phi_1(y)$ field, etc. We can analyze the contribution due to a single vertex by looking at the possible pairings of

$$\ldots\phi_1(x)\phi_1(x)\phi_2(x)\phi_2(x)\phi_1(y_1)\phi_1(y_2)\phi_2(y_3)\phi_2(y_4)\ldots \tag{22}$$

where y_i represent other spacetime points that are not x.

There are 4 equivalent ways of pairing since we have two pairs of identical fields from the x vertex. These four terms in the sum over pairings are then identical and can be summed to give a factor of 4. This factor of 4 is associated with the x vertex and a similar factor can arise from other vertices where all fields are paired with fields of other spacetime points. This 4 can be absorbed into a single factor of $-ig$ to get the Feynman rule



$$= -4ig(2\pi)^4\delta^4\left(\sum p_i\right) \tag{23}$$

3 Scalar QED

Now, we tackle scalar QED. We only care about the derivative coupling, so even though this action is not gauge invariant, take the action as:

$$\begin{aligned} I[\phi,\phi^\dagger,a_\mu] &= I_0[\phi,\phi^\dagger,a_\mu] + I_1[\phi,\phi^\dagger,a_\mu] \\ I_0[\phi,\phi^\dagger,a_\mu] &= \int d^4x\left[-\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\alpha(\partial_\mu a^\mu)^2\right] \\ I_1[\phi,\phi^\dagger,a_\mu] &= \int d^4x\left[iea_\mu\left((\partial^\mu\phi^\dagger)\phi - \phi^\dagger\partial^\mu\phi\right)\right]\end{aligned} \tag{24}$$

To calculate the vertex factor of a vertex with legs off-shell and not due to external fields, we can once again consider the vacuum-to-vacuum amplitude at order N in the coupling constant e. We can write this as

$$\begin{aligned}\langle \text{VAC, out}|\text{VAC, in}\rangle_N = \frac{(-e)^N}{N!}\prod_{i=1}^{N}\left(\int d^4x_i\right)\int\left(\prod_x d\phi(x)\prod_x d\phi^\dagger(x)\prod_{x,\mu} da_\mu\right) \\ \times\left(\prod_{i=1}^{N}a_{\mu_i}(x_i)\left((\partial^\mu\phi^\dagger(x_i))\phi(x_i) - \phi^\dagger(x_i)\partial^\mu\phi(x_i)\right)\right)e^{iI_0[\phi]}\end{aligned} \tag{25}$$

If we separate out the kinetic part of the action into a photon and complex scalar portion

$$\begin{aligned} I_0[\phi,\phi^\dagger,a_\mu] &= I_0^\phi[\phi,\phi^\dagger] + I_0^a[a_\mu] \\ I_0^\phi[\phi,\phi^\dagger] &= \int d^4x \left[-\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi\right] \\ I_0^a[a_\mu] &= \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\alpha(\partial_\mu a^\mu)^2\right], \end{aligned} \tag{26}$$

then we can separate the vacuum-to-vacuum amplitude into a product of two path integrals

$$\begin{aligned} \langle\text{VAC, out}|\text{VAC, in}\rangle_N = \frac{(-e)^N}{N!}\prod_{i=1}^N\left(\int d^4x_i\right)\int\left(\prod_{x,\mu}da_\mu\right)\prod_{i=1}^N a_{\mu_i}(x_i)e^{iI_0^a[a_\mu]} \\ \times\int\left(\prod_x d\phi(x)\prod_x d\phi^\dagger(x)\right) \\ \times\left(\prod_{i=1}^N\left((\partial^\mu\phi^\dagger(x_i))\phi(x_i) - \phi^\dagger(x_i)\partial^\mu\phi(x_i)\right)\right)e^{iI_0^\phi[\phi,\phi^\dagger]} \end{aligned} \tag{27}$$

It is clear from this form that the path integral over a_μ will just give a bunch of photon propagators and nothing new to the vertex factor, so we can factor this portion out and consider what's left, the ϕ-only amplitude

$$\begin{aligned} \langle\text{VAC, out}|\text{VAC, in}\rangle_N^\phi \equiv \frac{(-e)^N}{N!}\prod_{i=1}^N\left(\int d^4x_i\right)\int\left(\prod_x d\phi(x)\prod_x d\phi^\dagger(x)\right) \\ \times\left(\prod_{i=1}^N\left((\partial^\mu\phi^\dagger(x_i))\phi(x_i) - \phi^\dagger(x_i)\partial^\mu\phi(x_i)\right)\right)e^{iI_0^\phi[\phi,\phi^\dagger]} \end{aligned} \tag{28}$$

from which any extra factors besides e may contribute to the vertex factor.

To proceed, we need to multiply all the vertex contributions out. In general, the result will look like a sum of $2N$ terms, each a product of k A-like vertices and $N-k$ B-like vertices, for $0 \le k \le N$, where k varies between terms. We define the A- and B-like vertices by:

$$\begin{aligned} V_A^{\mu_i}(x_i) &= (\partial^\mu\phi^\dagger(x_i))\phi(x_i) \\ V_B^{\mu_i}(x_i) &= -\phi^\dagger(x_i)\partial^\mu\phi(x_i) \end{aligned} \tag{29}$$

It is important to note that the above splitting into terms is not the same splitting into terms as the real scalar field with derivative coupling. In that case, a sum of terms arose from the different pairings of fields in the path integral. We now have $2N$ terms and each term has its own path integral.

However, it should come as no surprise that the $2N$ terms naturally split into N subgroups of 2 terms. For every term, there will be another term in the sum that is completely identical except one vertex in the product, call it $V_A^{\mu'}(x')$, will be exchanged with $V_B^{\mu'}(x')$, evaluated at the same (singled-out) spacetime point x'. We will derive the vertex factor from considering the sum of these two terms. We are allowed to do this since x' is not unique, so we can do the same analysis for each possible vertex. Also, once we find the vertex factor for each of the N pairs of terms, we can continue to factor out similar terms to recover the product of vertices in Equation 28, but now knowing which vertex factor is associated with which of the vertices.

Let us now massage the A- and B-vertices into a more useful form. We can introduce delta functions and integrate by parts to get

$$\begin{aligned} V_A^{\mu_i}(x_i) &= -\int d^4y_i\frac{\partial}{\partial(x_i)_{\mu_i}}\delta^4(x_i-y_i)\phi(x_i)^\dagger\phi(y_i) \\ V_B^{\mu_i}(x_i) &= \int d^4y_i\frac{\partial}{\partial(y_i)_{\mu_i}}\delta^4(x_i-y_i)\phi(x_i)^\dagger\phi(y_i) \end{aligned} \tag{30}$$

A general term in our sum of $2N$ terms will look like

$$\int \left(\prod_{i=1}^{N} d^4 y_i\right) \left(\prod_{j\in G_A} -\frac{\partial}{\partial(x_j)_{\mu_j}}\delta^4(x_j - y_j)\right) \left(\prod_{k\in G_B} \frac{\partial}{\partial(y_k)_{\mu_k}}\delta^4(x_k - y_k)\right) \left(\prod_{i=1}^{N} \phi^\dagger(x_i)\phi(y_i)\right) \quad (31)$$

where G_A and G_B are the sets of indices for this term corresponding to A- and B-like vertices respectively. One important thing to note at this stage is that we could pull the product of fields out of all $2N$ terms. This lets us do the path integral in Equation 28. This implies that all $2N$ terms are multiplied by the *same* sum over pairings of complex scalar propagators.

As mentioned above, the general subgroup of two terms for our chosen x' will look something like

$$\int \left(\prod_{i=1}^{N} d^4 y_i\right) (\text{All other terms}) \left(-\frac{\partial}{\partial(x')_{\mu'}}\delta^4(x' - y') + \frac{\partial}{\partial(y')_{\mu'}}\delta^4(x' - y')\right) \left(\prod_{i=1}^{N} \phi^\dagger(x_i)\phi(y_i)\right) \quad (32)$$

where "All other terms" represents the other derivatives of delta functions for spacetime points that are not x', y'.

Plugging this into Equation 28 in place of the product over all full vertices, we get what I'll call the "reduced" ϕ-only amplitude

$$\begin{aligned}\langle \text{VAC, out}|\text{VAC, in}\rangle_N^{\phi,\text{ red.}} \equiv \frac{(-e)^N}{N!} &\left(\prod_{i=1}^{N}\int d^4x_i d^4y_i\right)\int\left(\prod_x d\phi(x)\prod_x d\phi^\dagger(x)\right)\\ &\times (\text{All other terms})\left(-\frac{\partial}{\partial(x')_{\mu'}}\delta^4(x'-y') + \frac{\partial}{\partial(y')_{\mu'}}\delta^4(x'-y')\right)\\ &\times \left(\prod_{i=1}^{N}\phi^\dagger(x_i)\phi(y_i)\right) e^{iI_0^\phi[\phi,\phi^\dagger]}\end{aligned} \quad (33)$$

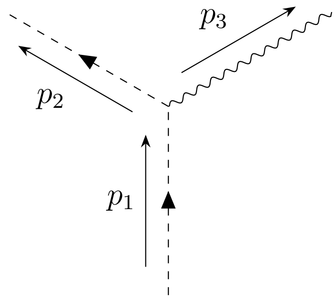
The last line we can easily do using what we saw from Section 2. Namely, it is a sum over all pairings of real scalar propagators, except each propagator has an additional multiplicative factor ϵ_{ab} that is 1 only when pairing a field ϕ with its conjugate $\phi^\dagger$, and 0 otherwise:

$$\begin{aligned}\langle \text{VAC, out}|\text{VAC, in}\rangle_N^{\phi,\text{ red.}} \equiv \frac{(-e)^N}{N!} &\left(\prod_{i=1}^{N}\int d^4x_i d^4y_i\right)\int\left(\prod_x d\phi(x)\prod_x d\phi^\dagger(x)\right)\\ &\times (\text{All other terms})\left(-\frac{\partial}{\partial(x')_{\mu'}}\delta^4(x'-y') + \frac{\partial}{\partial(y')_{\mu'}}\delta^4(x'-y')\right)\\ &\times \sum_{\text{pairings}}\prod_{\text{pairs}}\left[-i\mathscr{D}^{-1}\right]\end{aligned} \quad (34)$$

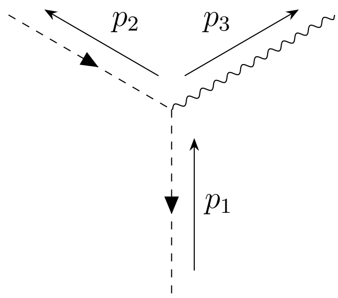
Next, the sum over pairings represents a unique Feynman diagram. Each term in this sum is a product of N disconnected propagators that only differ by which points are the "source" and "destination" points, as mentioned in Section 1. We then do integration by parts, and this has two effects. First, we get an overall sign from moving the derivative. This cancels with one copy of the -1 in $(-e)^N$. Second, we pull down a $\pm ip^{\mu'}$ from whichever propagators have x' and y', where the $\pm$ is for if x' or y' was a destination or source point, respectively. After this, we can then do the integral over y' to collapse the two disjoint propagators onto one another, and if we factor back in the photon propagator that depends on x', doing the x' integral gives a momentum conserving delta function and a factor of $(2\pi)^4$.

There is one more thing to consider before we write down the vertex factors. Specifically, the legs of a given vertex are distinct due to the scalar field being complex. We need to know what momentum to assign to each leg. As Weinberg mentions in Chapter 6, the field adjoint is associated with the point y while the field is associated with point x in $-i\Delta(x, y)$ which is the propagator from point y to x. In Equation 34, x' is associated with a field adjoint and y' associated with a field. Therefore, the momentum pulled down by the x' derivative is associated with the leg carrying charge away from the vertex, and the momentum pulled down by the y' derivative is associated with the leg carrying charge towards the vertex.

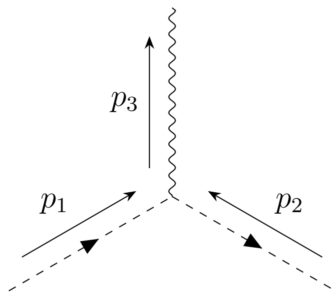
The Feynman rules are then



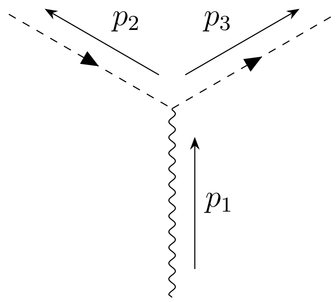
$$= ie(+p_2 + p_1)^\mu (2\pi)^4 \delta^4 \left(\sum p_i\right) \tag{35}$$



$$= ie(-p_1 - p_2)^\mu (2\pi)^4 \delta^4 \left(\sum p_i\right) \tag{36}$$



$$= ie(-p_2 + p_1)^\mu (2\pi)^4 \delta^4 \left(\sum p_i\right) \tag{37}$$



$$= ie(+p_3 - p_2)^\mu (2\pi)^4 \delta^4 \left(\sum p_i\right) \tag{38}$$

Once again, these hold for momenta p_i off the mass-shell, and one can check with Srednicki that these are the same as if they were calculated by treating the external legs as on-shell.

4 Feynman Rules for a General Derivative Coupling

This section isn't a formal proof (indeed this entire note is not a 100% rigorous proof), but there are some patterns that I expect to hold for a general interaction that involves derivatives of fields.

What we've seen so far is that using the path-integral method, we can rewrite an arbitrary field derivative as

$$\partial_\mu \psi_l(x) = -\int d^4y \psi_l(y) \frac{\partial}{\partial y^\mu} \delta^4(x - y) \tag{39}$$

which allows us to move a field, not a field derivative, into the path-integral. What this buys us is that the propagators are our usual propagators between field $\psi_l(x)$ and not field-derivatives (here we assume that $\psi_l(x)$ is not a derivative of some other field). When doing it this way, the spacetime point y will only ever appear in two locations, in the derivative of the delta function, and the field $\psi_l(y)$. Therefore, after we get a sum of paired propagators from the path integral, we can move the derivative back onto the propagator. The position-space propagator must be a Fourier transform, so the derivative then pulls down a factor of momentum. That momentum will *always* be associated with the derivative from the spacetime point y though. In this way, we can claim that this factor of momentum is indeed part of the vertex factor of the interaction vertex.

I have seen other texts calculate the vertex factor Feynman rules from considering the correlator

$$\langle 0| T\{\psi_l(x)\psi_m(y)\ldots\} |0\rangle \tag{40}$$

where the $\psi_l, \psi_m, \ldots$ are not composed of derivatives, and these fields are precisely the ones found in the interaction (but in the interaction they may have a derivative on them). This has always puzzled me. If our interaction has field derivatives must we not also consider correlators with field derivatives? If the legs are on-shell, this shouldn't matter, since whatever extra factors we get from considering a field derivative are compensated by the different field coefficients $u_l(\mathbf{p})$ we use in the LSZ reduction formula to take the legs on-shell. However, for off-shell legs, we are stuck with multiple possible correlators that differ by factors of momenta.

What the path integral tells us above is that we are allowed to consider such a correlator in Equation 40 since the $\psi_l, \psi_m, \ldots$ stand in for the fields that have already been stripped of their derivatives by integration by parts. It is in this sense that we actually *must* use the fields without derivatives in the correlator, since if we used a derivative of a field, we would have extra factors that would belong to a different vertex, and not the one we are considering when finding the vertex factor.

So, my heuristic formula for the Feynman rules is

1. For a theory of fields $\Psi_l^{(1)}(x)$, $\Psi_l^{(2)}(y)$, etc. and their derivatives, each field, field-adjoint pair has a single propagator given by the two-point correlator

$$-i\Delta_{lm}^{(i)}(x,y) = \langle 0| T\{\Psi_l^{(i)}(x)\Psi_m^{(i)\dagger}(y)\} |0\rangle \tag{41}$$

2. For an interaction vertex with coupling constant g and n_1 fields $\Psi_l^{(1)}(x)$, possibly including their derivatives, n_2 fields $\Psi_l^{(2)}(x)$, possibly including their derivatives, etc. The vertex factor can be found by calculating the order-g contribution to the correlator

$$\langle 0| T\{\Psi_{l_1}^{(1)}(x_1)\Psi_{l_2}^{(1)}(x_2)\ldots\Psi_{m_2}^{(2)}(y_2)\ldots\} \tag{42}$$

and stripping away the external propagators found from step 1. What's leftover is the vertex factor.