Isospin

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Introduction

I decided to write a note due to not being able to follow a derivation in class, and also struggling with finding correct information in textbooks. I don't think this is "huge" or honestly that remarkable, but it confused me for a few days, mainly because quite a few sources are contradictory. The issue comes from needing to distinguish carefully between fields and single-particle states. I'll give a brief motivation of the issue.

Consider the following interaction Lagrangian:

$$\mathscr{L}_{\text{int}} = \frac{g}{2}\bar{N}\vec{\sigma}\cdot\vec{\pi}N \tag{1}$$

where N is the proton/neutron double $N = (p \;\; n)$, σ_i are the Pauli matrices, and π^i are a set of real scalar fields. This is a first-pass model of pion-nucleon interactions, but doesn't work due to pions being parity odd as well as the chiral Lagrangian requiring derivative couplings of the pion field. Still, it works for illustrative purposes. Notice that this is invariant under a global $SU(2)$ transformation (that we'll prove in the next section).

The π^i do not create or annihilate particles of definite isospin (specifically, third component of isospin). So, in my class, and also in Schwartz section 22.3, new operators are defined

$$\pi^{\pm} \equiv \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2) \tag{2}$$

that create and annihilate particles of $I_3 = +1$ and $I_3 = -1$, respectively. $\pi^0 \equiv \pi^3$ too. As we'll see later, this sort-of makes sense when we consider:

$$I_3\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\\ 0\end{bmatrix} = \begin{bmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\\ 0\end{bmatrix} = +\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\\ 0\end{bmatrix} \tag{3}$$

The issue with this definition becomes evident when we multiply out the interaction term (neglecting the prefactors):

$$\bar{N}\vec{\sigma}\cdot\vec{\pi}N = \begin{bmatrix}\bar{p} & \bar{n}\end{bmatrix}\begin{bmatrix}\pi^0 & \sqrt{2}\pi^-\\ \sqrt{2}\pi^+ & -\pi^0\end{bmatrix}\begin{bmatrix}p\\ n\end{bmatrix} = \pi^0\bar{p}p - \pi^0\bar{n}n + \sqrt{2}\pi^-\bar{p}n + \sqrt{2}\pi^+\bar{n}p \tag{4}$$

Focusing on the third term, we see it corresponds to an interaction vertex where a neutron and a π^- enter the diagram and a proton exits. This manifestly violates isospin conservation, since the initial state has $I_3 = -3/2$ and the proton has $I_3 = +1/2$. We haven't included a coupling to the photon, but $\pi^{\pm}$ has charge ± 1, and this would also violate charge conservation.

Global $SU(2)$ Invariant Couplings

Before proceeding to resolving the isospin conservation problem, it's useful just to write out how the interaction is invariant under global $SU(2)$. To do so, let's generalize the interaction. Let ψ_l represent some multiplet of particles, and let the free Lagrangian be invariant under transformations

$$\psi_l(x) \to \psi_l + i\epsilon^\alpha (t_\alpha)_l{}^m \psi_m(x) \tag{5}$$

The t_α are the generators of transformations of some Lie group in some representation that the ψ_l is in. Next, define ϕ^γ to be in the adjoint representation, so that the respective free Lagrangian is invariant under

$$\phi^\gamma(x) \to \phi^\gamma(x) + \epsilon^\beta C^\gamma{}_{\alpha\beta} \phi^\alpha(x) \tag{6}$$

where the $C^\gamma{}_{\alpha\beta}$ are the real structure constants of the Lie algebra

$$[t_\alpha, t_\beta] = iC^\gamma{}_{\alpha\beta} t_\gamma \tag{7}$$

The reality condition on $C^\gamma{}_{\alpha\beta}$ is actually somewhat restrictive, but it guarantees that the generators are Hermitian.

The adjoint of the ψ field transforms like (and dropping the position labels)

$$(\psi^l)^\dagger \to (\psi^l)^\dagger - i\epsilon^\alpha (\psi^m)^\dagger (t_\alpha)_m{}^l \tag{8}$$

where we used the Hermiticity of the generators $(t_\alpha)_l{}^{m*} = (t_\alpha)_m{}^l$.

Let's see how the following object transforms to first order in ϵ (and also dropping position dependence for brevity):

$$\begin{aligned}(\psi^l)^\dagger (t_\alpha)_l{}^m \psi_m \to (\psi^l)^\dagger (t_\alpha)_l{}^m \psi_m &+ i\epsilon^\beta (\psi^l)^\dagger (t_\alpha)_l{}^m (t_\beta)_m{}^n \psi_n \\ &- i\epsilon^\beta (\psi^n)^\dagger (t_\beta)_n{}^l (t_\alpha)_l{}^m \psi_m \\ = (\psi^l)^\dagger (t_\alpha)_l{}^m \psi_m &+ i\epsilon^\beta (\psi^l)^\dagger [t_\alpha, t_\beta]_l{}^m \psi_m \\ = (\psi^l)^\dagger (t_\alpha)_l{}^m \psi_m &- \epsilon^\beta C^\gamma{}_{\alpha\beta} (\psi^l)^\dagger (t_\gamma)_l{}^m \psi_m \end{aligned} \tag{9}$$

So, effectively what's going on is a simultaneous transformation of the ψ multiplet and its adjoint is the same as transforming the generators coupling the two multiplets.

Next, let's see how the following quantity transforms

$$\begin{aligned}(\psi^l)^\dagger \phi^\alpha (t_\alpha)_l{}^m \psi_m \to (\psi^l)^\dagger \phi^\alpha (t_\alpha)_l{}^m &- \epsilon^\beta (\psi^l)^\dagger \phi^\alpha C^\gamma{}_{\alpha\beta} (t_\gamma)_l{}^m \psi_m \\ &+ \epsilon^\beta (\psi^l)^\dagger \phi^\gamma C^\alpha{}_{\gamma\beta} (t_\alpha)_l{}^m \psi_m \\ = (\psi^l)^\dagger \phi^\alpha (t_\alpha)_l{}^m & \end{aligned} \tag{10}$$

So this quantity is a scalar under global transformations. In particular,

$$\bar{N} \frac{\vec{\sigma}}{2} \cdot \vec{\pi} N \tag{11}$$

is invariant under global $SU(2)$ transformations.

One can generalize this depending on what the components of ψ_l are. For instance, if each ψ_l is itself a Dirac spinor, one can imagine coupling the spinor indices via gamma matrices, but this will not affect the structure of the coupling between the multiplet components.

The Fields Transform Contravariantly

The title says the resolution to the earlier isopsin conservation issue. I'll try to elaborate this.

When we consider a multiplet in the adjoint representation, we are considering a vector $\vec{\pi}$, where

$$\vec{\pi}(x) = \begin{bmatrix} \pi^1(x) \\ \pi^2(x) \\ \pi^3(x) \end{bmatrix} = \pi^1(x) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \pi^2(x) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \pi^3(x) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \pi^1(x)\hat{\mathbf{e}}_1 + \pi^2(x)\hat{\mathbf{e}}_2 + \pi^3(x)\hat{\mathbf{e}}_3 \tag{12}$$

where I've defined a basis

$$\hat{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{13}$$

which I'll call the adjoint basis. This name is due to the fact that the adjoint representation of a Lie algebra is unique in that it induces a natural basis since the relation $(t_\alpha)^\beta{}_\gamma = -iC^\beta{}_{\gamma\alpha}$ gives the *components* of the generator.

Next, we want to define our single-particle pion states as eigenstates of isospin, specifically of I^2 and I_3, where I_3 is given by

$$I_3 = -i(t_3)^i{}_j = -i\epsilon_{ij3} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

The eigenvectors with eigenvalue $+1, 0, -1$ are given by

$$|1,+1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \equiv \hat{\mathbf{e}}_+, \quad |1,0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \equiv \hat{\mathbf{e}}_0, \quad |1,-1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \equiv \hat{\mathbf{e}}_- \tag{15}$$

These eigenvectors are a perfectly good basis, so let's express our adjoint basis in terms of this new basis. One can check that the correct relation is

$$\hat{\mathbf{e}}_1 = \frac{1}{2}(\hat{\mathbf{e}}_+ + \hat{\mathbf{e}}_-), \quad \hat{\mathbf{e}}_2 = \frac{-i}{2}(\hat{\mathbf{e}}_+ - \hat{\mathbf{e}}_-), \quad \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_0 \tag{16}$$

Our expression for the $\vec{\pi}$ vector in the new basis is then

$$\vec{\pi}(x) = \frac{1}{\sqrt{2}} \left(\pi^1(x) - i\pi^2(x)\right) \hat{\mathbf{e}}_+ + \pi^3(x)\hat{\mathbf{e}}_0 + \frac{1}{\sqrt{2}} \left(\pi^1(x) + i\pi^2(x)\right) \hat{\mathbf{e}}_- \tag{17}$$

which makes it natural to identify the pion *fields* in this new basis as

$$\pi^\pm \equiv \frac{1}{\sqrt{2}} \left(\pi^1 \mp i\pi^2\right), \quad \pi^0 \equiv \pi^3 \tag{18}$$

What happened here? Well, our pion fields formed the coefficients of a vector, and our pion isospin states are a basis of this vector space. If we want to change the basis to a new basis, the basis will of course transform covariantly, while the coefficients transform contravariantly such that the overall vector is invariant. So, **the fields don't transform like the states under a change of basis**.

A Sanity Check

It makes sense that we associate, for instance, the field $\pi^+(x)$ with the positive isospin pion, but are we sure that $\pi^+(x)$ annihilates a positive isospin pion and creates its antiparticle? We can perform a brief sanity check. Consider a single-pion state. The full state vector is a tensor product between a vector in the isospin space and a vector that transforms in an irreducible representation of the Lorentz group. The pion has spin 0, so the full state vector has a momentum label as well as a label for the isospin space. Let's say our single-pion is in a definite state in the adjoint basis. We can rewrite such a state as a creation operator acting on the vacuum:

$$\left|\mathbf{p}, \pi^1\right\rangle \equiv |\mathbf{p}\rangle \otimes \hat{\mathbf{e}}_1 = a^\dagger(\mathbf{p}, \pi^1)\,|0\rangle \tag{19}$$

and similarly

$$\left|\mathbf{p}, \pi^2\right\rangle \equiv |\mathbf{p}\rangle \otimes \hat{\mathbf{e}}_2 = a^\dagger(\mathbf{p}, \pi^2)\,|0\rangle \tag{20}$$

Now, following the previous section, define the following linear combinations

$$\left|\mathbf{p}, \pi^\pm\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\mathbf{p}, \pi^1\right\rangle \pm i\left|\mathbf{p}, \pi^2\right\rangle\right) \tag{21}$$

These states have momentum $\mathbf{p}$ and are definite eigenstates of I_3 with isospin ± 1, respectively. We can factor out the creation and annihilation operators to rewrite this as

$$\left|\mathbf{p}, \pi^\pm\right\rangle = \frac{1}{\sqrt{2}}\left(a^\dagger(\mathbf{p}, \pi^1) \pm i a^\dagger(\mathbf{p}, \pi^2)\right)|0\rangle \tag{22}$$

which lets us identify

$$a^\dagger(\mathbf{p}, \pi^\pm) = \frac{1}{\sqrt{2}}\left(a^\dagger(\mathbf{p}, \pi^1) \pm i a^\dagger(\mathbf{p}, \pi^2)\right). \tag{23}$$

The corresponding annihilation operator would then be

$$a(\mathbf{p}, \pi^\pm) = \frac{1}{\sqrt{2}}\left(a(\mathbf{p}, \pi^1) \mp i a(\mathbf{p}, \pi^2)\right). \tag{24}$$

Working in the interaction picture, we want the field that annihilates $\pi^\pm$ and creates its antiparticle to be proportional to $a(\mathbf{p}, \pi^\pm)$. It's clear that we then need to take the linear combination of fields

$$\begin{aligned}
\pi^\pm(x) &= \frac{1}{\sqrt{2}}\left(\pi^1(x) \mp i\pi^2(x)\right) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_\mathbf{p}}}\left(\left(\frac{a(\mathbf{p}, \pi^1) \mp i a(\mathbf{p}, \pi^2)}{\sqrt{2}}\right) e^{ip\cdot x} + \left(\frac{a^\dagger(\mathbf{p}, \pi^1) \mp i a^\dagger(\mathbf{p}, \pi^2)}{\sqrt{2}}\right) e^{-ip\cdot x}\right) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_\mathbf{p}}}\left(a(\mathbf{p}, \pi^\pm) e^{ip\cdot x} + a^\dagger(\mathbf{p}, \pi^\mp) e^{-ip\cdot x}\right)
\end{aligned} \tag{25}$$

This a) confirms what we found in the last section, and b) confirms that the $\pi^\pm$ are each other's antiparticles.