S-Matrix Conventions

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Introduction

The purpose of this document is to help clarify different conventions between Weinberg and Peskin and Schroeder concerning the S-matrix and related quantities.

Weinberg

Weinberg uses the metric $(+,+,+,-)$. For calculating rates and cross sections, he defines the connected S-matrix element as:

$$\boxed{S_{\beta\alpha} = -2i\pi\delta^4(p_\beta - p_\alpha)M_{\beta\alpha}} \tag{1}$$

where $M_{\beta\alpha}$ is the delta-function free matrix element. As a reminder, connected here means that "no subset of particles in the state β (other than the whole state itself) has precisely the same four-momentum as some corresponding subset of particles in the state α."

An important note is necessary here:

$$\boxed{\text{Weinberg's Feynman rules are used to calculate } S_{\beta\alpha}} \tag{2}$$

The single-particle decay rate is given as

$$d\Gamma(\alpha \to \beta) = 2\pi|M_{\beta\alpha}|^2\delta^4(p_\beta - p_\alpha)d\beta \tag{3}$$

where $d\beta$ is the product of all $d^3\mathbf{p}$ for particles in β. In the initial particle center of mass frame with two particles in the final state, this simplifies to

$$\boxed{\frac{d\Gamma}{d\Omega} = \frac{2\pi k' E_1' E_2'}{E}|M_{\beta\alpha}|^2} \tag{4}$$

where we enforce $\mathbf{p}_1 = -\mathbf{p}_2$ and $k' = |\mathbf{p}_1'|$ that satisfied $E_1' + E_2' = E$ with E the initial particle energy.

The differential cross section for two particles in the initial state is given as

$$d\sigma(\alpha \to \beta) = \frac{(2\pi)^4}{u_\alpha}|M_{\beta\alpha}|^2\delta^4(p_\beta - p_\alpha)d\beta \tag{5}$$

where u_α the relative velocity given by

$$u_\alpha = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2} = \frac{|\mathbf{p}_1|(E_1 + E_2)}{E_1 E_2} \tag{6}$$

where the last equality is in the center of mass frame only.

In the center of mass frame with two particles in the final state, the differential cross section is given as

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{(2\pi)^4 k' E_1' E_2' E_1 E_2}{k E^2} |M_{\beta\alpha}|^2} \tag{7}$$

where $k' = |\mathbf{p}_1'| = |\mathbf{p}_2'|$ and $k = |\mathbf{p}_1| = |\mathbf{p}_2|$ such that $E_1 + E_2 = E_1' + E_2' = E$ is satisfied.

Peskin and Schroeder

Peskin and Schroeder use the metric $(+, -, -, -)$. Peskin and Schroeder define the S matrix as

$$S = \mathbf{1} + iT \tag{8}$$

where iT is the connected part of the S matrix. They then define the corresponding matrix element for two particles in the initial state as

$$\langle \mathbf{p}_1 ... \mathbf{p}_n | iT | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle = (2\pi)^4 \delta^4 (p_\mathcal{A} + p_\mathcal{B} - \sum p_f) \cdot i\mathcal{M} \tag{9}$$

where they call $\mathcal{M}$ the invariant matrix element. The above equation is rarely of any use though, since

$$\boxed{\text{Peskin and Schroeder's Feynman rules are used to calculate } i\mathcal{M}} \tag{10}$$

The decay rate is given as

$$\boxed{d\Gamma = \frac{1}{2m_\mathcal{A}} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}(m_\mathcal{A} \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)}(p_\mathcal{A} - \sum p_f)} \tag{11}$$

where the product term in parantheses with the product over phase space volume elements is the Lorentz Invariant Phase Space factor (LISP). This isn't given in the text, but for two particles in the final state, this reduces down to

$$\frac{d\Gamma}{d\Omega} = \frac{1}{(2\pi)^2} \frac{k}{8m_\mathcal{A}^2} |\mathcal{M}(m_\mathcal{A} \to p_1, p_2)|^2 \tag{12}$$

The cross section is given as

$$d\sigma = \frac{1}{2E_\mathcal{A} E_\mathcal{B} |v_\mathcal{A} - v_\mathcal{B}|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}(p_\mathcal{A}, p_\mathcal{B} \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)}(p_\mathcal{A} + p_\mathcal{B} - \sum p_f) \tag{13}$$

which reduces in the center of mass frame with two particles in the final state to:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{2E_\mathcal{A} E_\mathcal{B} |v_\mathcal{A} - v_\mathcal{B}|} \frac{|\mathbf{p}_1|}{(2\pi)^4 4 E_\text{cm}} |\mathcal{M}(p_\mathcal{A}, p_\mathcal{B} \to p_1, p_2)|^2} \tag{14}$$

Peskin and Schroeder don't give a nice formula for the relative velocity mainly because they always set it equal to 1 or 2 in the relativistic limit, although one can just use Weinberg's definition.