Selected Solutions to Weinberg's TQTF

Cameron Poe, Joey Takach, William Urdahl

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1 Conventions

We will follow all the same conventions as Weinberg does. Most importantly, a generic four vector is written as $A^\mu = (A^1, A^2, A^3, A^0)$, and the metric $\eta^{\mu\nu} = \mathrm{diag}(1, 1, 1, -1)$.

2 Chapter 2

2.1 Problem 2.1 – Quantum Lorentz Transformations for a Massive Particle

We will make use of equation (2.5.23) for how a massive particle state $\Psi_{p,\sigma}$ transforms under a homogenous Lorentz transformation $U(\Lambda)$:

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(W(\Lambda, p))\Psi_{\Lambda p,\sigma'} \tag{1}$$

The most difficult part of this problem is finding what the little group transformation W is. W is given by equation (2.5.10):

$$W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p) \tag{2}$$

Before we compute W, we can note two properties it must have. Since the little group for massive particles is $SO(3)$, we know that W, a representation of the little group, must be a rotation matrix. The other property is the rotation matrix must be a rotation about the x-axis. This is because $\mathbf{p}$ is in the y-direction, and therefore the boost $L(p)$ preserves four-vectors' x-components. Similarly, the boost Λ is in the z-direction and preserves x-components. The boost $L^{-1}(\Lambda p)$ boosts in the y- and z-directions, and must also preserve x-components. So the rotation W must leave x-components invariant, which means the rotation must be about the x-axis.

The energy of the W-boson in observer $\mathcal{O}$'s frame is $E = \sqrt{p^2 + m^2}$, and therefore the four-momentum is

$$p^\mu = (0, p, 0, E) \tag{3}$$

We will use equation (2.5.24) to calculate $L(p)$ and $L^{-1}(\Lambda p)$. The Lorentz factor to go from k^μ to p^μ is $\gamma = \frac{E}{m}$, so $\sqrt{\gamma^2 - 1} = \frac{p}{m}$. The components of the unit three-momentum are $\hat{p}_1 = \hat{p}_3 = 0$ and $\hat{p}_2 = 1$. The boost $L(p)$ is then

$$L(p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{E}{m} & 0 & \frac{p}{m} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{p}{m} & 0 & \frac{E}{m} \end{bmatrix} \tag{4}$$

Since $\mathcal{O}'$ is moving at speed v in the $+z$-direction relative to $\mathcal{O}$, the boost that takes us from $\mathcal{O}$ to $\mathcal{O}'$ is

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -v\gamma \\ 0 & 0 & -v\gamma & \gamma \end{bmatrix} \tag{5}$$

where $\gamma = \frac{1}{\sqrt{1-v^2}}$. Note that this γ is not referring to the gamma used to previously find $L(p)$, but rather refers to the boost from $\mathcal{O}$ to $\mathcal{O}'$.

The four-momenta to $\mathcal{O}'$ is

$$(\Lambda p)^\mu = (0, p, -v\gamma E, \gamma E) \tag{6}$$

The boost $L^{-1}(\Lambda p)$ is the inverse of $L(\Lambda p)$, and therefore boosts a particle with four-momentum $(\Lambda p)^\mu$ back into its rest frame. This is equivalent to boosting the particle in the opposite direction it was originally boosted in, so $L^{-1}(\Lambda p) = L(-\Lambda p)$. The Lorentz factor for this boost is $\gamma = \frac{E'}{m} = \frac{\gamma E}{m}$. The expression for L^i_0 can be simplified when solving for these components:

$$L^i_0(p) = \hat{p}_i\sqrt{\gamma^2 - 1} = \frac{p_i}{|\mathbf{p}|}\frac{|\mathbf{p}|}{m} = \frac{p_i}{m} \tag{7}$$

Therefore

$$L^1_0(-\Lambda p) = 0 \tag{8}$$
$$L^2_0(-\Lambda p) = -\frac{p}{m} \tag{9}$$
$$L^3_0(-\Lambda p) = \frac{v\gamma E}{m} \tag{10}$$

The boost $L^{-1}(\Lambda p)$ then reads

$$L^{-1}(\Lambda p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\gamma E}{m}\left(\frac{p^2+v^2\gamma mE}{p^2+v^2\gamma^2E^2}\right) & \frac{v\gamma pE}{m}\left(\frac{m-\gamma E}{p^2+v^2\gamma^2E^2}\right) & -\frac{p}{m} \\ 0 & \frac{v\gamma pE}{m}\left(\frac{m-\gamma E}{p^2+v^2\gamma^2E^2}\right) & \frac{v^2\gamma^3E^3+mp^2}{m(p^2+v^2\gamma^2E^2)} & \frac{v\gamma E}{m} \\ 0 & -\frac{p}{m} & \frac{v\gamma E}{m} & \frac{\gamma E}{m} \end{bmatrix} \tag{11}$$

Plugging all of this into Equation 2 gives the full little group element

$$W(\Lambda, p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\gamma m+E}{m+\gamma E} & \frac{v\gamma p}{m+\gamma E} & 0 \\ 0 & -\frac{v\gamma p}{m+\gamma E} & \frac{\gamma m+E}{m+\gamma E} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

We should note that $W(\Lambda, p)$ has the predicted form of a rotation matrix about the x-axis, where we identify $\cos(\theta) = \frac{\gamma m+e}{m+\gamma E}$ and $\sin(\theta) = \frac{v\gamma p}{m+\gamma E}$.

Since the W-boson is a spin-1 particle, the representation $D^{(j=1)}_{\sigma'\sigma}$ of $W(\Lambda, p)$ is simply a rotation matrix for 3D vectors, so we can immediately identify

$$D^{(1)}_{\sigma'\sigma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\gamma m+E}{m+\gamma E} & \frac{v\gamma p}{m+\gamma E} \\ 0 & -\frac{v\gamma p}{m+\gamma E} & \frac{\gamma m+E}{m+\gamma E} \end{bmatrix} \tag{13}$$

with subsequent rows and columns numbered -1, 0, and 1.

We are now able to write the full transformed state

$$\begin{aligned} U(\Lambda)\Psi_{p,+1} &= \sqrt{\frac{\gamma E}{E}} \sum_{\sigma'} D^{(1)}_{\sigma',+1}(W(\Lambda,p))\Psi_{\Lambda p,\sigma'} \\ &= \sqrt{\gamma}\left(D^{(1)}_{-1,+1}\Psi_{\Lambda p,-1} + D^{(1)}_{0,+1}\Psi_{\Lambda p,0} + D^{(1)}_{+1,+1}\Psi_{\Lambda p,+1}\right) \\ &= \sqrt{\gamma}\left(\frac{v\gamma p}{m+\gamma E}\Psi_{\Lambda p,0} + \frac{\gamma m+E}{m+\gamma E}\Psi_{\Lambda p,+1}\right) \\ \Aboxed{U(\Lambda)\Psi_{p,+1} &= \frac{\sqrt{\gamma}}{m+\gamma E}\left(v\gamma p\Psi_{\Lambda p,0} + (\gamma m+E)\Psi_{\Lambda p,+1}\right)} \end{aligned} \tag{14}$$

We can further check that when $v \to 0$, we get that $U(\Lambda)\Psi_{p,+1} = \Psi_{p,+1}$, as expected.

2.2 Problem 2.2 – Quantum Lorentz Transformations for a Massless Particle

To solve this problem, we do something similar to 2.1, where we find the little group element W, and read off the necessary values from this matrix to find $U(\Lambda)$.

The quantum Lorentz transformation for a massless particle is given by equation (2.5.42) in Weinberg:

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}}e^{i\sigma\theta(\Lambda,p)}\Psi_{\Lambda p,\sigma} \tag{15}$$

Weinberg equation 2.5.43 gives the relation between W, W as a Wigner rotation, and the special form of W for massless particles:

$$W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p) = S(\alpha(\Lambda,p),\beta(\Lambda,p))R(\theta(\Lambda,p)) \tag{16}$$

where we have the following relations:

$$S(\alpha,\beta) = \begin{bmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1-\xi & \xi \\ \alpha & \beta & -\xi & 1+\xi \end{bmatrix} \tag{17}$$

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

$$L(p) = R(\hat{\mathbf{p}})B(|\mathbf{p}|/\kappa) \tag{19}$$

$$B(u) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{u^2+1}{2u} & \frac{u^2-1}{2u} \\ 0 & 0 & \frac{u^2-1}{2u} & \frac{u^2+1}{2u} \end{bmatrix} \tag{20}$$

where $L(p)$ is boosting $k^\mu = (0,0,\kappa,\kappa)$ to p^μ, $R(\hat{\mathbf{p}})$ rotates the three-axis to the direction of the three-momentum $\hat{\mathbf{p}}$. Also, $\xi = \frac{1}{2}(\alpha^2+\beta^2)$.

The four-momentum of the photon to observer $\mathcal{O}$ is $p^\mu = (0,p,0,p)$. We boost to observer $\mathcal{O}'$ moving in the $+z$-direction with velocity $\beta = v$. The four-momentum to $\mathcal{O}'$ is then

$$\Lambda p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -v\gamma \\ 0 & 0 & -v\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ p \\ 0 \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ p \\ -v\gamma p \\ \gamma p \end{bmatrix} \tag{21}$$

with $\gamma = 1/\sqrt{1-v^2}$.

For $L(p)$, we wish to rotate $\mathbf{k}$, the standard massless three-momentum which is along the three-axis, into the $\hat{\mathbf{p}}$ direction. This is the $+y$-direction in our case. This is a rotation of $-\pi/2$ about the one-axis. Weinberg describes other rotation conventions that would give the same result (and also change the phase of one-particle states), but we will go with this as it only requires one rotation.[1] The rotation matrix then looks like:

$$R(\hat{\mathbf{p}}) = R_1(-\pi/2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\sin(-\pi/2) & 0 \\ 0 & \sin(-\pi/2) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

[1] See Hagimoto for the other convention.

The full $L(p)$ is then

$$L(p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{(p/\kappa)^2+1}{2(p/\kappa)} & \frac{(p/\kappa)^2-1}{2(p/\kappa)} \\ 0 & 0 & \frac{(p/\kappa)^2-1}{2(p/\kappa)} & \frac{(p/\kappa)^2+1}{2(p/\kappa)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{(p/\kappa)^2+1}{2(p/\kappa)} & \frac{(p/\kappa)^2-1}{2(p/\kappa)} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{(p/\kappa)^2-1}{2(p/\kappa)} & \frac{(p/\kappa)^2+1}{2(p/\kappa)} \end{bmatrix} \tag{23}$$

To find $L^{-1}(\Lambda p)$, first let $p' = \Lambda p$. We then note

$$L^{-1}(p') = (R(\hat{\mathbf{p}}')B(|\mathbf{p}'|/\kappa))^{-1} = B^{-1}(|\mathbf{p}'|/\kappa)R^{-1}(\hat{\mathbf{p}}') \tag{24}$$

Solving for $R(\hat{\mathbf{p}}')$, we can once again do a simple rotation about the one-axis, although this time it won't be a nice angle like $\pi/2$. Since this is a massless particle, we must have $|\mathbf{p}'| = E = \gamma p$, so $-\hat{\mathbf{p}}' = (0, -1/\gamma, v)$. Geometrically, we find that $\cos\theta = -v$ and $\sin\theta = -1/\gamma$. Our inverse rotation matrix then just sends $\cos\theta \to \cos\theta$ and $\sin\theta \to -\sin\theta$

$$R^{-1}(\hat{\mathbf{p}}') = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -v & -1/\gamma & 0 \\ 0 & 1/\gamma & -v & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

The inverse of $B(u)$ is not as simply as letting $u \to -u$ everywhere, but since the $(3,0)$ and $(0,3)$ elements correspond to $\beta\gamma$, we just put a minus on these since we are boosting in the opposite direction, i.e. in $-\beta$, so

$$B^{-1}(|\mathbf{p}'|/\kappa) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{(\gamma p/\kappa)^2+1}{2(\gamma p/\kappa)} & -\frac{(\gamma p/\kappa)^2-1}{2(\gamma p/\kappa)} \\ 0 & 0 & -\frac{(\gamma p/\kappa)^2-1}{2(\gamma p/\kappa)} & \frac{(\gamma p/\kappa)^2+1}{2(\gamma p/\kappa)} \end{bmatrix} \tag{26}$$

Our full $L(p')$ is:

$$L^{-1}(p') = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -v & -1/\gamma & 0 \\ 0 & \frac{p}{2\kappa} + \frac{\kappa}{2p\gamma^2} & -\frac{v\kappa^2+p^2\gamma^2}{2p\gamma\kappa} & -\frac{p\gamma}{2\kappa} + \frac{\kappa}{2p\gamma} \\ 0 & -\frac{p}{2\kappa} + \frac{\kappa}{2p\gamma^2} & -\frac{pv\gamma}{2\kappa} - \frac{v\kappa}{2p\gamma} & \frac{p\gamma}{2\kappa} + \frac{\kappa}{2p\gamma} \end{bmatrix} \tag{27}$$

The full little group element is then

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-v\kappa}{p} & \frac{v\kappa}{p} \\ 0 & \frac{v\kappa}{p} & 1 - \frac{v^2\kappa^2}{2p^2} & \frac{v^2\kappa^2}{2p^2} \\ 0 & \frac{v\kappa}{p} & -\frac{v^2\kappa^2}{2p^2} & 1 + \frac{v^2\kappa^2}{2p^2} \end{bmatrix} \tag{28}$$

Let's now match this W to the one in terms of θ, α, and β.

$$W = \begin{bmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1-\xi & \xi \\ \alpha & \beta & -\xi & 1+\xi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & -\alpha & \alpha \\ -\sin\theta & \cos\theta & -\beta & \beta \\ \alpha\cos\theta - \beta\sin\theta & \beta\cos\theta + \alpha\sin\theta & 1-\xi & \xi \\ \alpha\cos\theta - \beta\sin\theta & \beta\cos\theta + \alpha\sin\theta & -\xi & 1+\xi \end{bmatrix} \tag{30}$$

The top left 2-by-2 block let's us easily read off what θ is, while the top right 2-by-2 block let's us read α and β. We see that $\theta = 0$, $\alpha = 0$, and $\beta = v\kappa/p$. Therefore, the transformed photon state is

$$\boxed{U(\Lambda)\Psi_{p,+1} = \sqrt{\gamma}\Psi_{\Lambda p,+1}} \tag{31}$$

2.3 Problem 2.3 – Galilean Group Commutation Relations

To begin, we define our inhomogeneous Galilean transformations as a pair (G, a), where G is a 4x4 matrix acting on 4-vectors, and a is a 4-vector shift representing the translation elements. The group law is the same as the Poincare case: $(\bar{G}, \bar{a})(G, a) = (\bar{G}G, \bar{G}a + \bar{a})$. The analogues to the Lorentz condition here are two-fold; both the spatial norm of vectors on the same time slice, and the time coordinate of any vector must be preserved under arbitrary (homogeneous) Galilean transformations G. These conditions have the form

$$\sum_i (Gx)_i (Gy)_i = \sum_i x_i y_i \quad \text{and} \quad (Gx)_0 = x_0 \ . \tag{32}$$

In component notation, these have the form

$$G_{ik}G_{jk} = \delta_{ij} \quad \text{and} \quad G_{0\alpha} = \delta_{0\alpha} \ . \tag{33}$$

From here on, we will use latin indices to refer to spatial components (1,2,3) and greek indices for arbitrary components. We will also write the sums implicitly unless the notation is wack.

These conditions tell us that the spatial block of the matrix G must be an ordinary 3x3 rotation matrix R, and that the bottom row of the matrix is null save for the time component. The boost components (G_{i0}) are unconstrained, so we set them as the vector $\mathbf{v}$. In block-diagonal notation, G then takes the form:

$$G = \begin{bmatrix} R & \mathbf{v} \\ 0 & 1 \end{bmatrix} \ . \tag{34}$$

We will be interested in the form of the infinitesimal transformations, eg when we approximately have $G = 1 + \omega$ (or $G_{\alpha\beta} = \delta_{\alpha\beta} + \omega_{\alpha\beta}$), where ω is a matrix representing the infinitesimal parameters. Concretely,

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} r & \mathbf{b} \\ 0 & 0 \end{bmatrix} = 1 + \omega \tag{35}$$

where r is our infinitesimal rotation, and $\mathbf{b}$ our infinitesimal boost.

Applying the Galilean conditions to this infinitesimal form, we obtain

$$\begin{gathered} \delta_{ij} = G_{ik}G_{jk} = (\delta_{ik} + \omega_{ik})(\delta_{jk} + \omega_{jk}) = \delta_{ij} + \omega_{ij} + \omega_{ji} + \mathcal{O}(\omega^2) \\ \Rightarrow \omega_{ij} = -\omega_{ji} \end{gathered} \tag{36}$$

$$\delta_{0\alpha} = G_{0\alpha} = \delta_{0\alpha} + \omega_{0\alpha} + \mathcal{O}(\omega^2) \quad \Rightarrow \quad \omega_{0\alpha} = 0 \; . \tag{37}$$

This just confirms what we already knew, namely that the infinitesimal matrix r is antisymmetric, and the bottom row of the infinitesimal matrix ω is 0.

As with the relativistic case, there are 10 independent generators: 3 for rotations, 3 for boosts, and 4 for spacetime translations. When we represent these group elements on the Hilbert space as $U(G, a)$, we can then expand the infinitesimal elements as:

$$U(1 + \omega, \varepsilon) = 1 + \frac{i}{2} r_{ij} J_{ij} + i b_i K_i - i\varepsilon_i P_i - i\varepsilon_0 P_0 \; . \tag{38}$$

The factor of $1/2$ again comes from the fact that since r is antisymmetric, we can also choose the generators J_{ij} in our representation to be antisymmetric, so the double sum over i, j doubles the contribution from each independent parameter.

Following the procedure in Section 2.4, we can expand the product $U(G, a)U(1+\omega, \varepsilon)U^{-1}(G, a)$ in two different ways; one where we first apply the group law, and one where we first expand. However, there is a key difference to the Galilean case from the Lorentz case. The representation of the Galilean group acting on the Hilbert space is intrinsically projective, and moreover, the central charges cannot be eliminated by a redefinition of the commutators (see Section 2.7). This means that the group law in the representation will have the form:

$$U(\bar{G}, \bar{a})U(G, a) = \exp\Big[i\phi(\bar{G}, \bar{a}; G, a)\Big] U(\bar{G}G, \bar{G}a + \bar{a}) \; . \tag{39}$$

However, the discussion in Section 2.7 shows that this extra phase is equivalent to the inclusion of central charges in the commutation relations, so we will ignore the phase for now, and manually add in the central charges once we have the basic commutators.

We need to expand both sides of the equation:

$$U(G, a)U(1 + \omega, \varepsilon)U^{-1}(G, a) = U(1 + G\omega G^{-1}, G\varepsilon - G\omega G^{-1}a) \; . \tag{40}$$

Expanding the RHS is made much easier through use of the block diagonal matrices for G, ω written above. We have:

$$\begin{aligned} 1 + \frac{i}{2}(RrR^T)_{mn} J_{mn} &+ i(R\mathbf{b} - RrR^T\mathbf{v})_m K_m \\ &- i\Big[(R(\boldsymbol{\varepsilon} + a_0\mathbf{b}) + RrR^T(a_0\mathbf{v} - \mathbf{a}) - \varepsilon_0\mathbf{v})_m P_m + \varepsilon_0 P_0\Big] \; . \end{aligned} \tag{41}$$

And for the left side, we have:

$$U(G, a)\Big[1 + \frac{i}{2} r_{ij} J_{ij} + i b_i K_i - i\varepsilon_i P_i - i\varepsilon_0 P_0\Big] U^{-1}(G, a) \; . \tag{42}$$

Expanding out the RHS making sure the indices on the infinitesimal parameters are the same as those on the LHS, we can match the coefficients of those parameters on either side to arrive at the transformation laws for each of the generators:

$$\begin{aligned} U(G, a)J_{ij}U^{-1}(G, a) &= R_{mi}R_{nj}\Big(J_{mn} - 2K_m v_n - 2P_m(a_0 v_n - a_n)\Big) \\ U(G, a)K_i U^{-1}(G, a) &= R_{mi}(K_m - a_0 P_m) \\ U(G, a)P_i U^{-1}(G, a) &= R_{mi}P_m \\ U(G, a)P_0 U^{-1}(G, a) &= P_0 + v_m P_m \; . \end{aligned} \tag{43}$$

Now, treating G, a as infinitesimal transformations of the Galilean group, we can expand the U's to first order, eg for the transformation of P_i we get:

$$(\delta_{mi} + r_{mi})P_m = \Big(1 + \frac{i}{2} r_{mn} J_{mn} + i b_m K_m - i\varepsilon_m P_m - i\varepsilon_0 P_0\Big) P_i \Big(1 - \frac{i}{2} r_{mn} J_{mn} - i b_m K_m + i\varepsilon_m P_m + i\varepsilon_0 P_0\Big) \tag{44}$$

which, when cancelling out the P_i terms, leads to

$$r_{mn}\delta_{ni}P_m = \frac{i}{2}r_{mn}[J_{mn},P_i] + ib_m[K_m,P_i] - i\varepsilon_m[P_m,P_i] - i\varepsilon_0[P_0,P_i] \ . \tag{45}$$

Equating terms with common infinitesimal coefficients (and remembering r_{mn}, J_{mn} are anti-symmetric), we get for the commutators:

$$i[J_{mn},P_i] = \delta_{ni}P_m - \delta_{mi}P_n \quad , \quad [K_m,P_i] = [P_m,P_i] = [P_0,P_i] = 0 \ . \tag{46}$$

With the convention $J_{12} \equiv J_3$, etc, we can do the same with the other transformation laws to get all of the commutators:

$$\begin{aligned}
&[J_i,J_j] = i\epsilon_{ijk}J_k, \quad [J_i,K_j] = i\epsilon_{ijk}K_k, \quad [K_i,K_j] = 0 \\
&[J_i,P_j] = i\epsilon_{ijk}P_k, \quad [K_i,P_j] = 0, \quad [K_i,P_0] = -iP_i \\
&[P_0,J_i] = [P_0,P_i] = [P_i,P_j] = 0 \ .
\end{aligned} \tag{47}$$

Note that these do not yet match the commutation relations that Weinberg writes on Pg. 62 for the Galilean group. This is because, as discussed before, the work we have done so far has assumed that the Galilean group has an ordinary representation on the Hilbert space, and not a projective one. This does not mean the work we have done is completely invalidated, but it does mean that we have to add to the commutation relations a central charge term proportional to the identity, so that the commutators in general go from:

$$[t_b,t_c] = iC^a{}_{bc}t_a \quad \Rightarrow \quad [t_b,t_c] = iC^a{}_{bc}t_a + iC_{bc}\mathbb{1}. \tag{48}$$

Thus, we rewrite our commutators above as:

$$\begin{aligned}
&[J_i,J_j] = i\epsilon_{ijk}J_k + iA_{ij}, \quad [J_i,K_j] = i\epsilon_{ijk}K_k + iB_{ij}, \quad [K_i,K_j] = iD_{ij} \\
&[J_i,P_j] = i\epsilon_{ijk}P_k + iE_{ij}, \quad [K_i,P_j] = iF_{ij}, \quad [K_i,P_0] = -iP_i + iL_{i0} \\
&[P_0,J_i] = iN_{0i}, \quad [P_0,P_i] = iQ_{0i}, \quad [P_i,P_j] = iR_{ij} \ .
\end{aligned} \tag{49}$$

To determine these constants C_{bc} in more detail, we need to apply the Jacobi identity to each of our commutators, which in general takes the form:

$$\Big[t_a,[t_b,t_c]\Big] + \Big[t_c,[t_a,t_b]\Big] + \Big[t_b,[t_c,t_a]\Big] = 0 \ . \tag{50}$$

As an example of how this works, consider now the Jacobi identity applied to P_0, K_i, P_j:

$$\begin{aligned}
0 &= \Big[P_0,[K_i,P_j]\Big] + \Big[P_j,[P_0,K_i]\Big] + \Big[K_i,[P_j,P_0]\Big] \\
&= [P_j, iP_i - iL_{i0}] = -i[P_i,P_j] = R_{ij} \\
&\Rightarrow R_{ij} = 0 \ .
\end{aligned} \tag{51}$$

In the first line, the inner commutators of the first and third terms give only central charges (constants), so the outer commutators vanish. In the second line, we used that the commutator of the two P_i generators gives another charge term, which we could then set to 0 because of the Jacobi identity. We now need to repeat this process for the other commutators. We have a few other similar cases:

P_0, J_i, J_j:

$$\begin{aligned}
0 &= \Big[P_0,[J_i,J_j]\Big] + \Big[J_j,[P_0,J_i]\Big] + \Big[J_i,[J_j,P_0]\Big] \\
&= i\epsilon_{ijk}[P_0,J_k] = -\epsilon_{ijk}N_{0k} \\
&\Rightarrow N_{0i} = 0 \ .
\end{aligned} \tag{52}$$

P_0, J_i, P_j:

$$\begin{aligned} 0 &= \Big[P_0, [J_i, P_j]\Big] + \Big[P_j, [P_0, J_i]\Big] + \Big[J_i, [P_j, P_0]\Big] \\ &= i\epsilon_{ijk}[P_0, P_k] = -\epsilon_{ijk}Q_{0k} \\ &\qquad\qquad \Rightarrow Q_{0i} = 0\ . \end{aligned} \tag{53}$$

J_i, K_j, K_k:

$$\begin{aligned} 0 &= \Big[J_i, [K_j, K_k]\Big] + \Big[K_k, [J_i, K_j]\Big] + \Big[K_j, [K_k, J_i]\Big] \\ &= i\epsilon_{ijm}[K_k, K_m] - i\epsilon_{ikm}[K_j, K_m] \\ &= -\epsilon_{ijm}D_{km} + \epsilon_{ikm}D_{jm} \end{aligned} \tag{54}$$

This is the first non-trivial one. First note that from (49), D_{ij} is antisymmetric by definition, so all diagonal elements vanish. Next, pluging into the above every possibility of $i = j \neq k$ yields $D_{ij} = 0$ for $i \neq j$. These combined mean $D_{ij} = 0$ uniformly.

Now we look at the central charges where the "base" commutation relations make use of the Levi-Civita symbol for their structure constants. Specifically, A_{ij}, B_{ij}, E_{ij}. Note how A_{ij} is antisymmetric by definition, but B_{ij}, E_{ij} must be shown to be so. For E_{ij}, we consider the Jacobi identity for P_0, J_i, K_j:

$$\begin{aligned} 0 &= \Big[P_0, [J_i, K_j]\Big] + \Big[K_j, [P_0, J_i]\Big] + \Big[J_i, [K_j, P_0]\Big] \\ &= i\epsilon_{ijk}[P_0, K_k] - i[J_i, P_j] \\ &= -\epsilon_{ijk}P_k + \epsilon_{ijk}L_{i0} + \epsilon_{ijk}P_k + E_{ij} \\ &= -\epsilon_{ijk}P_k + E_{ij}\ . \end{aligned} \tag{55}$$

Similarly, we can swap the indices on J_i and K_j to get the result $0 = -\epsilon_{jik}P_k + E_{ji} = \epsilon_{ijk}P_k + E_{ji}$. Adding this and the last line of (55) yields $E_{ij} + E_{ji} = 0$, so E_{ij} is antisymmetric. To show the same for B_{ij}, we look at the Jacobi identity for J_i, J_j, K_k:

$$\begin{aligned} 0 &= \Big[J_i, [J_j, K_k]\Big] + \Big[K_k, [J_i, J_j]\Big] + \Big[J_j, [K_k, J_i]\Big] \\ &= i\epsilon_{jkm}[J_i, K_m] - i\epsilon_{ijm}[J_m, K_k] - i\epsilon_{ikm}[J_j, K_m] \\ &= -\epsilon_{jkm}\epsilon_{imn}K_n + \epsilon_{ijm}\epsilon_{mkn}K_n + \epsilon_{ikm}\epsilon_{jmn}K_n \\ &\quad - \epsilon_{jkm}B_{im} + \epsilon_{ijm}B_{mk} + \epsilon_{ikm}B_{jm}\ . \end{aligned} \tag{56}$$

The first line of the last equality vanishes from properties of ϵ_{ijk} (also since the generators K_n are linearly independent to each other and to the identity on the lie algebra vector space, their coefficients must identically vanish if the RHS is equal to 0). The second line, when you plug in all cases of $i = k \neq j$, yields the desired antisymmetry condition, $B_{ij} + B_{ji} = 0$ (alternatively, if you're feeling particularly pretentious or feel so inclined as to sacrifice a whole sheet of paper to the algebra gods, you can do some tedious index gymnastics by contracting with various Levi-Civita tensors and Kronecker deltas to arrive at the same thing, but this method is *objectively* dumb and stupid).

Now that we know A_{ij}, B_{ij}, E_{ij} are all antisymmetric, we can move to redefine the corresponding generators to eliminate the central charges. Because these charges are antisymmetric, we can write them in terms of the antisymmetric Levi-Civita tensor, eg:

$$A_{ij} = \epsilon_{ijk}a_k, \quad B_{ij} = \epsilon_{ijk}b_k, \quad E_{ij} = \epsilon_{ijk}e_k\ , \tag{57}$$

for some sets of undetermined constants a_k, b_k, e_k. We then redefine the generators in the following way:

$$\tilde{J}_i \equiv J_i + a_i, \quad \tilde{K}_i \equiv K_i + b_i, \quad \tilde{P}_i \equiv P_i + e_i\ . \tag{58}$$

With these redefined generators, the commutation relations are rid of the A, B, E charges, eg:

$$[\tilde{J}_i, \tilde{J}_j] = [J_i, J_j] = i\epsilon_{ijk}J_k + iA_{ij} = i\epsilon_{ijk}(\tilde{J}_k - a_k) + i\epsilon_{ijk}a_k = i\epsilon_{ijk}\tilde{J}_k \ . \tag{59}$$

The same holds with the other commutation relations. From here on, we will work with the $\tilde{J}$ generators, but refer to them without tildes as J.

We can summarize our results so far by writing the commutation relations:

$$\begin{aligned} &[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = 0 \\ &[J_i, P_j] = i\epsilon_{ijk}P_k, \quad [K_i, P_j] = iF_{ij}, \quad [K_i, P_0] = -iP_i + iL_{i0} \\ &[P_0, J_i] = [P_0, P_i] = [P_i, P_j] = 0 \ . \end{aligned} \tag{60}$$

We still have two charges to deal with, L_{i0} and F_{ij}. We can show that $L_{i0} = 0$ with the Jacobi identity applied to P_0, J_i, K_j:

$$\begin{aligned} 0 &= \Big[P_0, [J_i, K_j]\Big] + \Big[K_j, [P_0, J_i]\Big] + \Big[J_i, [K_j, P_0]\Big] \\ &= i\epsilon_{ijk}[P_0, K_k] - i[J_i, P_j] \\ &= -\epsilon_{ijk}P_k + \epsilon_{ijk}L_{k0} + \epsilon_{ijk}P_k = \epsilon_{ijk}L_{k0} \\ &\qquad\qquad \Rightarrow L_{i0} = 0 \ . \end{aligned} \tag{61}$$

Now we consider the final charge, F_{ij}. First we consider the Jacobi identity applied to J_i, P_j, K_k:

$$\begin{aligned} 0 &= \Big[J_i, [P_j, K_k]\Big] + \Big[K_k, [J_i, P_j]\Big] + \Big[P_j, [K_k, J_i]\Big] \\ &= i\epsilon_{ijm}[K_k, P_m] - i\epsilon_{ikm}[P_j, K_m] \\ &= -\epsilon_{ijm}F_{km} - \epsilon_{ikm}F_{mj} \ . \end{aligned} \tag{62}$$

As before, we can plug in all cases of $i = k \neq j$ to kill the second term and get that $F_{ij} = 0$ for all $i \neq j$, meaning all off-diagonal elements are 0. We can then plug in the cases where ijk is some permutation of even permutation of 123, which leads to the condition $F_{11} = F_{22} = F_{33}$, which when combined with the former condition means that F_{ij} is proportional to the identity. In other words, we may choose

$$F_{ij} = -M\delta_{ij} \ . \tag{63}$$

Because this central charge does not take the form of the structure constants contracted with some set of scalars (equation 2.7.10 of Weinberg), it follows that we cannot eliminate it by redefinition of the generators. This is not explicitly stated by Weinberg, but can be shown by assuming a redefinition of the generators $t_a \to \tilde{t}_a$ eliminates the central charges, and solving for the condition on the redefinition parameters.

To make contact with the Galilean group commutation relations presented by the late and great Steve, we make one final redefinition, namely

$$H \equiv P_0 + M \ , \tag{64}$$

where our P_0 is playing the role of W in the text. This doesn't change the commutation relations because M is proportional to the identity, and therefore commutes with everything. Once we formally enlarge our group by adding in this mass generator M, our final commutation relations become:

$$\boxed{\begin{aligned} &[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = 0 \\ &[J_i, P_j] = i\epsilon_{ijk}P_k, \quad [K_i, P_j] = -iM\delta_{ij}, \quad [K_i, P_0] = -iP_i \\ &[P_0, J_i] = [P_0, P_i] = [P_i, P_j] = 0 \\ &[M, t] = 0 \text{ for all generators } t \text{ in the Lie algebra.} \end{aligned}} \tag{65}$$

Note: Actually identifying M with the mass of the particles in the theory is another story, and beyond the scope of these solutions.

2.4 Problem 2.4 – Casimir Elements of O(3,1)

This problem is relatively straightforward, asking us to prove that $P^\mu P_\mu$ and $W^\mu W_\mu$, the Casimir elements of $O(3,1)$, do indeed commute with all the generators of the algebra. One could solve this directly by calculating the commutators with P^μ and $J^{\rho\sigma}$, but this is quite involved. An easier way is to show that these Casimir elements transform as scalars under inhomogeneous Lorentz transformation, i.e. $[P^\mu P_\mu, U(\Lambda, a)] = [W^\mu W_\mu, U(\Lambda, a)] = 0$, which implies for infinitiessimal parameters that these elements commute with all the generators. This is particularly easy for $P^\mu P_\mu$ since it transforms as a four-vector under inhomogeneous Lorentz transformations, Weinberg 2.4.9:

$$U(\Lambda, a)P^\mu U^{-1}(\Lambda, a) = \Lambda_\nu{}^\mu P^\nu \tag{66}$$

We therefore have

$$\begin{aligned}
U(\Lambda, a)P^\mu P_\mu U^{-1}(\Lambda, a) &= U(\Lambda, a)P^\mu U^{-1}(\Lambda, a)U(\Lambda, a)P_\mu U^{-1}(\Lambda, a) \\
&= \Lambda_\nu{}^\mu P^\nu \Lambda_{\rho\mu} P^\rho \\
&= (\Lambda^{-1})^\mu{}_\nu \Lambda^\rho{}_\mu P^\nu P_\rho \\
&= \delta^\rho{}_\nu P^\nu P_\rho \\
&= P^\rho P_\rho \\
&\implies U(\Lambda, a)P^\mu P_\mu = P^\mu P_\mu U(\Lambda, a) \\
&\implies [P^\mu P_\mu, U(\Lambda, a),] = 0
\end{aligned} \tag{67}$$

W is a little harder to work with. We first begin with seeing how W^μ transforms under inhomogeneous Lorentz transformations. A useful reminder is how $J^{\rho\sigma}$ transforms:

$$U(\Lambda, a)J^{\rho\sigma}U^{-1}(\Lambda, a) = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \left(J^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu\right) \tag{68}$$

Applying this to W^μ:

$$\begin{aligned}
U(\Lambda, a)W^\mu U^{-1}(\Lambda, a) &= \eta^{\mu\nu}\epsilon_{\nu\alpha\beta\gamma} U(\Lambda, a)J^{\alpha\beta}P^\gamma U^{-1}(\Lambda, a) \\
&= \eta^{\mu\nu}\epsilon_{\nu\alpha\beta\gamma} U(\Lambda, a)J^{\alpha\beta}U^{-1}(\Lambda, a)U(\Lambda, a)P^\gamma U^{-1}(\Lambda, a) \\
&= \eta^{\mu\nu}\epsilon_{\nu\alpha\beta\gamma}\Lambda_\rho{}^\alpha \Lambda_\sigma{}^\beta \Lambda_\tau{}^\gamma \left(J^{\rho\sigma} - a^\rho P^\sigma + a^\sigma P^\rho\right) P^\tau
\end{aligned} \tag{69}$$

Focusing on the second and third terms,

$$\begin{aligned}
&- \eta^{\mu\nu}\epsilon_{\nu\alpha\beta\gamma}\Lambda_\rho{}^\alpha \Lambda_\sigma{}^\beta \Lambda_\tau{}^\gamma a^\rho P^\sigma + \eta^{\mu\nu}\epsilon_{\nu\alpha\beta\gamma}\Lambda_\rho{}^\alpha \Lambda_\sigma{}^\beta \Lambda_\tau{}^\gamma a^\sigma P^\rho P^\tau \\
&= \eta^{\mu\nu}\left(-\epsilon_{\nu\alpha\beta\gamma}(\Lambda^{-1}a)^\alpha(\Lambda^{-1}P)^\beta(\Lambda^{-1}P)^\gamma + \epsilon_{\nu\alpha\beta\gamma}(\Lambda^{-1}a)^\beta(\Lambda^{-1}P)^\alpha(\Lambda^{-1}P)^\gamma\right) \\
&= \eta^{\mu\nu}\left(-\epsilon_{\nu\alpha\beta\gamma}(\Lambda^{-1}a)^\alpha(\Lambda^{-1}P)^\beta(\Lambda^{-1}P)^\gamma + \epsilon_{\nu\beta\gamma\alpha}(\Lambda^{-1}a)^\beta(\Lambda^{-1}P)^\gamma(\Lambda^{-1}P)^\alpha\right) \\
&= 0
\end{aligned} \tag{70}$$

where to arrive at the third line we did two permutations of the Levi-Cevita symbol $\epsilon_{\nu\alpha\beta\gamma} = \epsilon_{\nu\beta\gamma\alpha}$ and swapped $(\Lambda^{-1}P)^\alpha$ and $(\Lambda^{-1}P)^\gamma$ since they commute. We are left with

$$U(\Lambda, a)W^\mu U^{-1}(\Lambda, a) = \eta^{\mu\nu}\epsilon_{\nu\alpha\beta\gamma}\Lambda_\rho{}^\alpha \Lambda_\sigma{}^\beta \Lambda_\tau{}^\gamma J^{\rho\sigma}P^\tau \tag{71}$$

Before we proceed, we will need to recall some identities of the Levi-Cevita symbol. In four dimensions, we have the following identity:

$$\epsilon_{\mu\alpha\beta\gamma}\epsilon^{\mu\delta\epsilon\zeta} = \delta^{\delta\epsilon\zeta}_{\alpha\beta\gamma} \tag{72}$$

where the δ on the right is the generalized Kronecker delta. The definition of this symbol is

$$\delta^{\delta\epsilon\zeta}_{\alpha\beta\gamma} = \sum_{p\in \mathrm{P}} \mathrm{sgn}(p)\delta^{\delta}_{\alpha}\delta^{\epsilon}_{\beta}\delta^{\zeta}_{\gamma} \tag{73}$$

where P is the set of all possible permutations of either the top or bottom row. Therefore,

$$\epsilon_{\mu\alpha\beta\gamma}\epsilon^{\mu\delta\epsilon\zeta} = \delta^{\delta}{}_{\alpha}\delta^{\epsilon}{}_{\beta}\delta^{\zeta}{}_{\gamma} + \delta^{\zeta}{}_{\alpha}\delta^{\delta}{}_{\beta}\delta^{\epsilon}{}_{\gamma} + \delta^{\epsilon}{}_{\alpha}\delta^{\zeta}{}_{\beta}\delta^{\delta}{}_{\gamma} - \delta^{\delta}{}_{\alpha}\delta^{\zeta}{}_{\beta}\delta^{\epsilon}{}_{\gamma} - \delta^{\epsilon}{}_{\alpha}\delta^{\delta}{}_{\beta}\delta^{\zeta}{}_{\gamma} - \delta^{\zeta}{}_{\alpha}\delta^{\epsilon}{}_{\beta}\delta^{\delta}{}_{\gamma} \tag{74}$$

We now have

$$\begin{aligned} U(\Lambda,a)W^{\mu}W_{\mu}U^{-1}(\Lambda,a) &= U(\Lambda,a)W^{\mu}U^{-1}(\Lambda,a)U(\Lambda,a)W_{\mu}U^{-1}(\Lambda,a) \\ &= \epsilon^{\mu}{}_{\alpha\beta\gamma}\epsilon_{\mu\delta\epsilon\zeta}\Lambda_{\rho}{}^{\alpha}\Lambda_{\sigma}{}^{\beta}\Lambda_{\tau}{}^{\gamma}\Lambda_{\phi}{}^{\delta}\Lambda_{\chi}{}^{\epsilon}\Lambda_{\psi}{}^{\zeta}J^{\rho\sigma}P^{\tau}J^{\phi\chi}P^{\psi} \\ &= \epsilon^{\mu\alpha\beta\gamma}\epsilon_{\mu\delta\epsilon\zeta}\Lambda_{\rho}{}^{\alpha}\Lambda_{\sigma}{}^{\beta}\Lambda_{\tau}{}^{\gamma}\Lambda^{\phi}{}_{\delta}\Lambda^{\chi}{}_{\epsilon}\Lambda^{\psi}{}_{\zeta}J^{\rho\sigma}P^{\tau}J_{\phi\chi}P_{\psi} \end{aligned} \tag{75}$$

Focusing just on the Levi-Cevita and Λ portion

$$\begin{aligned} &\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\mu\delta\epsilon\zeta}\Lambda_{\rho}{}^{\alpha}\Lambda_{\sigma}{}^{\beta}\Lambda_{\tau}{}^{\gamma}\Lambda^{\phi}{}_{\delta}\Lambda^{\chi}{}_{\epsilon}\Lambda^{\psi}{}_{\zeta} \\ &\quad= (\delta^{\delta}{}_{\alpha}\delta^{\epsilon}{}_{\beta}\delta^{\zeta}{}_{\gamma} + \delta^{\zeta}{}_{\alpha}\delta^{\delta}{}_{\beta}\delta^{\epsilon}{}_{\gamma} + \delta^{\epsilon}{}_{\alpha}\delta^{\zeta}{}_{\beta}\delta^{\delta}{}_{\gamma} - \delta^{\delta}{}_{\alpha}\delta^{\zeta}{}_{\beta}\delta^{\epsilon}{}_{\gamma} - \delta^{\epsilon}{}_{\alpha}\delta^{\delta}{}_{\beta}\delta^{\zeta}{}_{\gamma} \\ &\quad- \delta^{\zeta}{}_{\alpha}\delta^{\epsilon}{}_{\beta}\delta^{\delta}{}_{\gamma})\Lambda_{\rho}{}^{\alpha}\Lambda_{\sigma}{}^{\beta}\Lambda_{\tau}{}^{\gamma}\Lambda^{\phi}{}_{\delta}\Lambda^{\chi}{}_{\epsilon}\Lambda^{\psi}{}_{\zeta} \\ &\quad= \Lambda_{\rho}{}^{\delta}\Lambda_{\sigma}{}^{\epsilon}\Lambda_{\tau}{}^{\zeta}\Lambda^{\phi}{}_{\delta}\Lambda^{\chi}{}_{\epsilon}\Lambda^{\psi}{}_{\zeta} + \Lambda_{\rho}{}^{\zeta}\Lambda_{\sigma}{}^{\delta}\Lambda_{\tau}{}^{\epsilon}\Lambda^{\phi}{}_{\delta}\Lambda^{\chi}{}_{\epsilon}\Lambda^{\psi}{}_{\zeta} + \Lambda_{\rho}{}^{\epsilon}\Lambda_{\sigma}{}^{\zeta}\Lambda_{\tau}{}^{\delta}\Lambda^{\phi}{}_{\delta}\Lambda^{\chi}{}_{\epsilon}\Lambda^{\psi}{}_{\zeta} \\ &\quad- \Lambda_{\rho}{}^{\delta}\Lambda_{\sigma}{}^{\zeta}\Lambda_{\tau}{}^{\epsilon}\Lambda^{\phi}{}_{\delta}\Lambda^{\chi}{}_{\epsilon}\Lambda^{\psi}{}_{\zeta} - \Lambda_{\rho}{}^{\epsilon}\Lambda_{\sigma}{}^{\delta}\Lambda_{\tau}{}^{\zeta}\Lambda^{\phi}{}_{\delta}\Lambda^{\chi}{}_{\epsilon}\Lambda^{\psi}{}_{\zeta} - \Lambda_{\rho}{}^{\zeta}\Lambda_{\sigma}{}^{\epsilon}\Lambda_{\tau}{}^{\delta}\Lambda^{\phi}{}_{\delta}\Lambda^{\chi}{}_{\epsilon}\Lambda^{\psi}{}_{\zeta} \\ &\quad= \delta^{\phi}{}_{\rho}\delta^{\chi}{}_{\sigma}\delta^{\psi}{}_{\tau} + \delta^{\psi}{}_{\rho}\delta^{\phi}{}_{\sigma}\delta^{\chi}{}_{\tau} + \delta^{\chi}{}_{\rho}\delta^{\psi}{}_{\sigma}\delta^{\phi}{}_{\tau} - \delta^{\phi}{}_{\rho}\delta^{\psi}{}_{\sigma}\delta^{\chi}{}_{\tau} - \delta^{\chi}{}_{\rho}\delta^{\phi}{}_{\sigma}\delta^{\psi}{}_{\tau} - \delta^{\psi}{}_{\rho}\delta^{\chi}{}_{\sigma}\delta^{\phi}{}_{\tau} \\ &\quad= \epsilon^{\mu\phi\chi\psi}\epsilon_{\mu\rho\sigma\tau} \end{aligned} \tag{76}$$

Therefore

$$\begin{aligned} U(\Lambda,a)W^{\mu}W_{\mu}U^{-1}(\Lambda,a) &= \epsilon^{\mu\phi\chi\psi}\epsilon_{\mu\rho\sigma\tau}J^{\rho\sigma}P^{\tau}J_{\phi\chi}P_{\psi} = W^{\mu}W_{\mu} \\ \implies U(\Lambda,a)W^{\mu}W_{\mu} &= W^{\mu}W_{\mu}U(\Lambda,a) \\ \implies [W^{\mu}W_{\mu}, U(\Lambda,a)] &= 0 \end{aligned} \tag{77}$$

So, since both $P^{\mu}P_{\mu}$ and $W^{\mu}W_{\mu}$ commute with an arbitrary $U(\Lambda,a)$, these operators must also commute with all generators of inhomogeneous Lorentz transformations, and are therefore Casimir elements.

2.5 Problem 2.5 – Massive Particles in 2+1 Dimensions

2.6 Problem 2.6 – Massless Particles in 2+1 Dimensions

We will proceed in a very similar fashion to Weinberg. Even though we are now in 2+1 dimensions, we will still use greek letters to label the three-vector components. We introduce the unit time-like and unit light-like vectors.

$$k^{\mu} = (0,1,1), \quad t^{\mu} = (0,0,1) \tag{78}$$

We introduce the little group elements W that is defined as the group of elements of O(3,1) that leave the light-like vector invariant, $(Wk)^{\mu} = k^{\mu}$. We have the following identities:

$$\begin{aligned} t^{\mu}t_{\mu} = -1 &\implies t^{\mu}t_{\mu} = (Wt)^{\mu}(Wt)_{\mu} = -1 \\ t^{\mu}k_{\mu} = -1 &\implies t^{\mu}k_{\mu} = (Wt)^{\mu}(Wk)^{\mu} = (Wt)^{\mu}k_{\mu} = -1 \end{aligned} \tag{79}$$

If we represent our transformed time-like vector as $(Wt)^\mu = (\beta, \zeta, \gamma)$, these two equations give constraints on the transformed time-like vector, and we are left with:

$$(Wt)^\mu = \left(\beta, \frac{\beta^2}{2}, 1 + \frac{\beta^2}{2}\right) = (\beta, \zeta, 1 + \zeta) \tag{80}$$

with $\zeta = \beta^2/2$.

One possible matrix that maps t^μ to $(Wt)^\mu$ is the boost given by

$$S^\mu{}_\nu(\beta) = \begin{bmatrix} 1 & -\beta & \beta \\ \beta & 1-\zeta & \zeta \\ \beta & -\zeta & 1+\zeta \end{bmatrix} \tag{81}$$

One can check that $S(\beta) \in SO^+(3,1)$, but we will not show that here. We now have

$$(St)^\mu = (Wt)^\mu \implies (S^{-1}Wt)^\mu = t^\mu \tag{82}$$

Since t^μ is left invariant by the operation $S^{-1}W$, this operation must be a rotation of the spatial coordinates. However, we can also show that

$$(Sk)^\mu = (Wk)^\mu \implies (S^{-1}Wk)^\mu = k^\mu \tag{83}$$

which means that if $S^{-1}W$ is a rotation, it must be a rotation about zero degrees, or the identity, so

$$W^\mu{}_\nu(\beta) = S^\mu{}_\nu(\beta) = \begin{bmatrix} 1 & -\beta & \beta \\ \beta & 1-\zeta & \zeta \\ \beta & -\zeta & 1+\zeta \end{bmatrix} \tag{84}$$

The little group is therefore one dimensional and isomorphic to the the real numbers $\mathbb{R}$ under addition.

The little group element for infinitiessimal β is

$$W^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu = \delta^\mu{}_\nu + \begin{bmatrix} 0 & -\beta & \beta \\ \beta & 0 & 0 \\ \beta & 0 & 0 \end{bmatrix} \tag{85}$$

where we discard ζ since it is quadratic in β. Contracting with the metric gives us

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & -\beta & \beta \\ \beta & 0 & 0 \\ -\beta & 0 & 0 \end{bmatrix} \tag{86}$$

The form of the little group element, $U(W)$, for infinitessimal β is simple. Weinberg equation (2.4.3) was derived without any explicit reference to the number of dimensions, so we can carry this formula as well as the later commutation relations (2.4.12-2.4.14) over wholesale to 2+1 dimensions. Our indices now range over 0 to 2, and we define the following generators:

$$J \equiv J^{12}, \quad K^1 \equiv J^{01}, \quad K^2 \equiv J^{02} \tag{87}$$

Weinberg equation (2.4.3) then reads

$$U(1+\omega) = 1 - i\beta J - i\beta K^1 = 1 + i\beta B \tag{88}$$

where $B \equiv -J - K^1$, and the P^1, P^2 generators are excluded just because their affect on the state Ψ is simply a phase, just like in the 3+1 case.

Since σ was just our label to denote all other degrees of freedom, σ thus labels the eigenvalues of the generator B. We then have the following transformation:

$$U(W)\Psi_{k,\sigma} = e^{i\beta B}\Psi_{k,\sigma} = e^{i\beta\sigma}\Psi_{k,\sigma} \tag{89}$$

which implies a D-matrix of:

$$D_{\sigma'\sigma}(W) = e^{i\beta\sigma}\delta_{\sigma'\sigma} \tag{90}$$

and a full transformed state as

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}}e^{i\beta(\Lambda)\sigma}\Psi_{\Lambda p,\sigma} \tag{91}$$

2.6.1 An Aside: Projective Representations of SO(2,1)

At this point however, we need to be quite careful. σ is not necessarily the helicity as we would imagine it for 3+1 dimensions for two reasons: There has been no restriction on the values σ could take and the generator B does not correspond solely to the 3-angular momentum. Since we do not have experimental evidence, it's not as easy to say σ must be a discrete value, so we must analyze the central charges and topology of $SO(2,1)$.

Since the central charges for $SO(3,1)$ made no reference to the dimensionality of the space, we can likewise conclude that all central charges for $SO(2,1)$ can be eliminated or the generators can be redefined to include the central charges, just like for $SO(3,1)$.

However, the topologies are quite different. We proceed like Weinberg. Let's encode an arbitrary three-vector V^μ as a matrix:

$$v = \begin{bmatrix} V^0+V^2 & V^1 \\ V^1 & V^0-V^2 \end{bmatrix} \tag{92}$$

Why do we do this? Well the determinant gives us the invariant length $V^\mu V_\mu$:

$$\det v = -(V^0)^2 + (V^1)^2 + (V^2)^2 = V^\mu V_\mu \tag{93}$$

We notice also that v is symmetric. Therefore, the transformation involving real matrices λ

$$v \to \lambda v \lambda^T \tag{94}$$

is also symmetric since

$$(\lambda v\lambda^T)^T = (v\lambda^T)\lambda^T = \lambda v^T\lambda^T = \lambda v\lambda^T \tag{95}$$

How does this transformation affect the determinant?

$$\det\left(\lambda v\lambda^T\right) = \det\lambda\det v\det\lambda^T = (\det\lambda)^2\det v \tag{96}$$

So, if $\det\lambda = \pm 1$, then $\det v$, the invariant length, is preserved. Let's now restrict our study to λ with $\det\lambda = \pm 1$. We can further restrict the λ we consider because this arbitrary phase, just like in the 3+1 case, can be chosen such that $\det\lambda = +1$.

Further, we can compose two transformations and see they obey the group transformation law:

$$(\lambda\bar{\lambda})v(\lambda\bar{\lambda})^T = \lambda(\bar{\lambda}v\bar{\lambda}^T)\lambda^T \tag{97}$$

So, the λ form a group, $SL(2,R)$. This group preserves the invariant length of a three-vector, the same effect as elements of $SO(2,1)$. We however note that λ and $-\lambda$ are different elements of $SL(2,R)$, but they nevertheless produce the same Lorentz transformation $\lambda v\lambda^T$, and therefore $SO(2,1)$ is not the same as $SL(2,R)$ but rather $SL(2,R)/Z_2$.

We now investigate the topology. We apply the polar decomposition theorem to an arbitrary real matrix λ:

$$\lambda = oe^s \tag{98}$$

where o is an orthogonal matrix and s is a symmetric matrix:

$$o^T o = 1, \quad s = s^T \tag{99}$$

From o being orthogonal, we must have $\det o = \pm 1$, however, we know $\det \lambda = +1$, so

$$\det \lambda = (\det o)(\det e^s) = (\det o)e^{\operatorname{Tr} s} \tag{100}$$

but $e^{\operatorname{Tr} s}$ is a positive number, so $\det o = +1$ for $\det \lambda$ to be positive. But, for $\det \lambda$ to be $+1$ exactly, we also require $\operatorname{Tr} s = 0$, so to summarize

$$\begin{aligned} \det o &= 1 \\ \operatorname{Tr} s &= 0 \end{aligned} \tag{101}$$

The o therefore belong to the group $SO(2)$, and the s are the group of symmetric traceless matrices. In particular, the o correspond to rotations in the spatial components. First note that $V^0 = \frac{1}{2} \operatorname{Tr} v$. If $s = 0$, then λ leaves V^0 invariant:

$$V^0 = \frac{1}{2} \operatorname{Tr} v \to \frac{1}{2} \operatorname{Tr} ovo^T = \frac{1}{2} Tr o^T ov = \frac{1}{2} Tr v = V^0 \tag{102}$$

so o is just a rotation.

Our o can be written in general as:

$$o = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \tag{103}$$

subject to the constraint

$$\det o = a^2 + b^2 = 1 \tag{104}$$

Which means the topology of $SO(2)$ is the same as S_1, the circle.

s, our 2-by-2 symmetric, traceless matrix can in general be written as:

$$s = \begin{bmatrix} c & d \\ d & -c \end{bmatrix} \tag{105}$$

with $c, d \in \mathbb{R}$, so the topology of this group is the same as two-dimensional flat space R_2. Since the polar decomposition theorem gurantees a unique decomposition, the group $SL(2, R)$ is just $R_2 \times S_1$. Now, the sign change of $+\lambda$ and $-\lambda$ can only be caused o since e^s is always positive, so the topology of $SO(2, 1)$ is $R_2 \times S_1/Z_2$.

It can be shown that the topology of S_1/Z_2 is isomorphic to S_1, and this topology is not simply connected. The fundamental group of S_1 is Z. Therefore, unlike with $SO(3, 1)$, we have no way to constrict the value σ takes. Recall that for $SO(3, 1)$, we were able to say that a rotation of 4π can be deformed to the identity, since it is composed of two rotations of 2π which is a double loop in S_3/Z_2. This meant that $e^{i4\pi\sigma} = 1$, which constrained σ to be integer or half-integer. However, for S_1/Z_2, since the topology is isomorphic to S_1, the classes of loops are labeled by winding number. For loops of non-zero winding number, we cannot deform these into the identity, and therefore, for non-zero β, we have no such identity of $e^{i\beta\sigma} = 1$. Our σ is unconstrained. Particles exhibiting this phenomenon have been described as "anyons," and have been studied in condensed matter. The moral of the story is σ can take any value.

Returning now to the problem, we will proceed with characterizing how the states change under P and T. The matrix $\mathscr{P}$ that acts on three-vectors is:

$$\mathscr{P} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{106}$$

A few things to note: $\mathscr{P}$ looks different from the 3+1D case since $\mathscr{P}$ must be in the

3 Chapter 3

3.1 Problem 3.1 – A Separable Interaction

In this problem, we are tasked to construct explicit expressions for the in/out states as well as the S-matrix given the matrix elements of the interaction potential:

$$(\Phi_\beta, V\Phi_\alpha) = g\; u_\beta\; u^*_\alpha \tag{107}$$

where

$$\sum_\alpha |u_\alpha|^2 = 1\ . \tag{108}$$

To do this, we use the Lippman-Schwinger equations expanded in a basis of free particle states:

$$\Psi^\pm_\alpha = \Phi_\alpha + \int d\beta \frac{T^\pm_{\beta\alpha}\Phi_\beta}{E_\alpha - E_\beta \pm i\varepsilon} \quad , \quad T^\pm_{\beta\alpha} \equiv (\Phi_\beta, V\Psi^\pm_\alpha)\ . \tag{109}$$

We can recursively expand the expression for the in/out states:

$$\Psi^\pm_\alpha = \Phi_\alpha + \int d\beta \frac{(\Phi_\beta, V\Phi_\alpha)\Phi_\beta}{E_\alpha - E_\beta \pm i\varepsilon} + \int d\beta \int d\gamma \frac{(\Phi_\beta, V\Phi_\gamma)(\Phi_\gamma, V\Phi_\alpha)\Phi_\beta}{(E_\alpha - E_\beta \pm i\varepsilon)(E_\alpha - E_\gamma \pm i\varepsilon)} + \ldots \tag{110}$$

then use (107) to rewrite this as:

$$\begin{aligned} \Psi^\pm_\alpha &= \Phi_\alpha + g\int d\beta \frac{u_\beta u^*_\alpha \Phi_\beta}{E_\alpha - E_\beta \pm i\varepsilon} + g^2 \int d\beta \int d\gamma \frac{u_\beta u^*_\gamma u_\gamma u^*_\alpha \Phi_\beta}{(E_\alpha - E_\beta \pm i\varepsilon)(E_\alpha - E_\gamma \pm i\varepsilon)} + \ldots \\ &= \Phi_\alpha + g\int d\beta \frac{u_\beta u^*_\alpha \Phi_\beta}{E_\alpha - E_\beta \pm i\varepsilon} \sum_{n=0}^{\infty} \left(g\int d\gamma \frac{|u_\gamma|^2}{E_\alpha - E_\gamma \pm i\varepsilon} \right)^n . \end{aligned} \tag{111}$$

The sum is now expressed as a geometric series. We can be sure that this series does indeed converge, because (108) guarantees that $|u_\gamma|^2$ dies off sufficiently fast at infinity to use contour integration and pick up poles in the energy from the denominators. We can then define this geometric ratio as $R^\pm_\alpha$, and evaluate the series as $(1 - R^\pm_\alpha)^{-1}$. Thus we have an explicit, non-recursive expression for the in/out states:

$$\Psi^\pm_\alpha = \Phi_\alpha + g\int d\beta \frac{u_\beta u^*_\alpha \Phi_\beta}{(E_\alpha - E_\beta \pm i\varepsilon)(1 - R^\pm_\alpha)}\ . \tag{112}$$

We can even remove the resolution of identity at this point to write

$$\boxed{\Psi^\pm_\alpha = \left[1 + (1 - R^\pm_\alpha)^{-1}(E_\alpha - H_0 \pm i\varepsilon)^{-1} V \right] \Phi_\alpha} \tag{113}$$

We can also find an explicit expression for the S-matrix elements $S_{\beta\alpha} = (\Psi^-_\beta, \Psi^+_\alpha)$:

$$\begin{aligned}S_{\beta\alpha} &= (\Phi_\beta, \Phi_\alpha) \\ &+ (\Phi_\beta, V(E_\beta - H_0 + i\varepsilon)^{-1}(1-R^+_\beta)^{-1}\Phi_\alpha) + (\Phi_\beta, (1-R^+_\alpha)^{-1}(E_\alpha - H_0 + i\varepsilon)^{-1}V\Phi_\alpha) \\ &+ (\Phi_\beta, V(E_\beta - H_0 + i\varepsilon)^{-1}(1-R^+_\beta)^{-1}(1-R^+_\alpha)^{-1}(E_\alpha - H_0 + i\varepsilon)^{-1}V\Phi_\alpha)\ .\end{aligned} \tag{114}$$

Since $(1 - R^+_\alpha)$ is a c-number, we can move it through the inner products with impunity, and then act the H_0 terms on the now adjacent eigenstates $\Phi_{\alpha/\beta}$ in the second line. We can then write the second line above as:

$$\begin{aligned}&(\Phi_\beta, V\Phi_\alpha)\left(\frac{1}{(1-R^+_\beta)(E_\beta - E_\alpha + i\varepsilon)} + \frac{1}{(1-R^+_\alpha)(E_\alpha - E_\beta + i\varepsilon)}\right) \\ &= gu_\beta u^*_\alpha\left(\frac{1}{(1-R^+_\beta)(E_\beta - E_\alpha + i\varepsilon)} + \frac{1}{(1-R^+_\alpha)(E_\alpha - E_\beta + i\varepsilon)}\right)\ .\end{aligned} \tag{115}$$

For the third line of (114), we can insert a complete set of states between the two H_0 terms, which gets us

$$\begin{aligned}&\frac{1}{(1-R^+_\beta)(1-R^+_\alpha)}\int d\gamma \frac{(\Phi_\beta, V\Phi_\gamma)(\Phi_\gamma, V\Phi_\alpha)}{(E_\beta - E_\gamma + i\varepsilon)(E_\alpha - E_\gamma + i\varepsilon)} \\ &= \frac{gu_\beta u^*_\alpha}{(1-R^+_\beta)(1-R^+_\alpha)}\int d\gamma \frac{g|u_\gamma|^2}{(E_\beta - E_\gamma + i\varepsilon)(E_\alpha - E_\gamma + i\varepsilon)}\ .\end{aligned} \tag{116}$$

We can then write the entire S-matrix element as

$$\boxed{\begin{aligned}S_{\beta\alpha} = \delta(\beta - \alpha) + \frac{gu_\beta u^*_\alpha}{(1-R^+_\beta)(1-R^+_\alpha)}&\left[\frac{1-R^+_\alpha}{E_\beta - E_\alpha + i\varepsilon} + \frac{1-R^+_\beta}{E_\alpha - E_\beta + i\varepsilon}\right. \\ &\left. + \int d\gamma \frac{g|u_\gamma|^2}{(E_\beta - E_\gamma + i\varepsilon)(E_\alpha - E_\gamma + i\varepsilon)}\right]\ .\end{aligned}} \tag{117}$$

This can potentially be simplified by combining the first two terms in brackets, but I am weary about the $i\varepsilon$ terms and the pole structure they represent.

3.2 Problem 3.2 – A Spin-1 Resonance

This question is a simple application of the formulae in chapter 3.7 on resonances. We start with the equation for the cross section of a 2-body channel n going into a 2-body channel n' written in the center-of-mass frame:

$$\sigma(n \to n'; E) = \frac{\pi(2j_R + 1)}{k^2(2s_1+1)(2s_2+1)}\frac{\Gamma_n\Gamma_{n'}}{(E-E_R)^2 + \Gamma^2/4} \tag{118}$$

Since we are looking at elastic scattering, $n = n'$. The cross section we are given is also at resonance, so $E = E_R$. Finally, at $\sqrt{s} = 150$ GeV, the electron and positron are ultrarelativistic, so $k = 75$ GeV. Plugging in the values for the spins and the cross section $\sigma(n \to n; E_R) = 10^{-34}\ \text{cm}^{-2} = 10^{-10}$ bn, we find:

$$\begin{aligned}\sigma(n \to n; E_R) &= \frac{\pi(2j_R + 1)}{k^2(2s_1+1)(2s_2+1)}\frac{\Gamma_n^2}{(E-E_R)^2 + \Gamma^2/4} \\ &= \frac{3\pi}{k^2}\left(\frac{\Gamma_n}{\Gamma}\right)^2 \\ &\implies \boxed{\Gamma_n/\Gamma = 0.0124}\end{aligned} \tag{119}$$

We also have:

$$\begin{aligned} &\sigma(n \to n'; E)/\sigma_{\text{total}}(n; E) = \Gamma_{n'}/\Gamma \\ \Longrightarrow &\boxed{\sigma_{\text{total}}(n; E_R) = 8.08 \text{ nb} = 8.08 \times 10^{-33} \text{ cm}^{-2}} \end{aligned} \tag{120}$$

3.3 Problem 3.7 – In/Out States For Separable Interactions

Our task here is to find explicit solutions for the in/out states defined by the Lippman-Schwinger equations

We are tasked with showing that the states $\Phi_{E\mathbf{p}j\sigma lsn}$ defined by equation (3.7.5)

$$\begin{aligned} (\Phi_{\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2 n'}, \Phi_{E\mathbf{p}j\sigma lsn}) = \sqrt{\frac{E}{|\mathbf{p}_1|E_1E_2}}\delta^3(\mathbf{p} - \mathbf{p}_1 - \mathbf{p}_2)\delta(E - E_1 - E_2)\delta_{n',n} \\ \times \sum_{m,\mu} C_{s_1s_2}(s, \mu; \sigma_1\sigma_2)C_{ls}(j, \sigma; m, \mu)Y_l^m(\hat{\mathbf{p}}_1) \end{aligned} \tag{121}$$

give the proper normalization in the center of mass frame (Weinberg 3.7.6):

$$(\Phi_{E'\mathbf{p}'j'\sigma'l's'n'}, \Phi_{E0j\sigma lsn}) = \delta^3(\mathbf{p}')\delta(E' - E)\delta_{j',j}\delta_{\sigma',\sigma}\delta_{l',l}\delta_{s',s}\delta_{n',n} \tag{122}$$

To show the validity of Equation 122, we proceed by working with the left hand side. Inserting a resolution of the identity in terms of states $\Phi_{\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2 n}$ gives

$$\begin{aligned} (\Phi_{E'\mathbf{p}'j'\sigma'l's'n'}, \Phi_{E0j\sigma lsn}) &= \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 \sum_{\sigma_1\sigma_2\bar{n}} (\Phi_{E'\mathbf{p}'j'\sigma'l's'n'}, \Phi_{\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2\bar{n}})\,(\Phi_{\mathbf{p}_1\sigma_1\mathbf{p}_2\sigma_2\bar{n}}, \Phi_{E0j\sigma lsn}) \\ &= \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 \sum_{\sigma_1\sigma_2\bar{n}} \frac{\sqrt{EE'}}{k_1E_1E_2}\delta^3(\mathbf{p}_1 + \mathbf{p}_2)\delta(E - E_1 - E_2)\delta_{\bar{n},n} \\ &\times \delta^3(\mathbf{p}' - \mathbf{p}_1 - \mathbf{p}_2)\delta(E' - E_1 - E_2)\delta_{\bar{n},n'} \sum_{m,\mu} C_{s_1s_2}(s, \mu; \sigma_1\sigma_2) \\ &\times C_{ls}(j, \sigma; m, \mu) \sum_{m',\mu'} C_{s_1s_2}(s', \mu'; \sigma_1\sigma_2)C_{l's'}(j', \sigma'; m', \mu')Y_l^m(\hat{\mathbf{p}}_1)Y_{l'}^{m'*}(\hat{\mathbf{p}}_1) \end{aligned} \tag{123}$$

where $k_i \equiv |\mathbf{p}_i|$ and $E_i \equiv \sqrt{k_i^2 + M_i^2}$.

We can rewrite the delta functions as

$$\begin{aligned} &\delta^3(\mathbf{p}_1 + \mathbf{p}_2)\delta(E - E_1 - E_2)\delta^3(\mathbf{p}' - \mathbf{p}_1 - \mathbf{p}_2)\delta(E' - E_1 - E_2) \\ &\quad = \delta^3(\mathbf{p}_1 + \mathbf{p}_2)\delta(E - E_1 - E_2)\delta^3(\mathbf{p}')\delta(E' - E) \end{aligned} \tag{124}$$

Let us now do the integral over $\mathbf{p}_2$. The only parts of Equation 123 that depend on $\mathbf{p}_2$ is then

$$\begin{aligned} &\int d^3\mathbf{p}_2 \frac{1}{E_2}\delta^3(\mathbf{p}_1 + \mathbf{p}_2)\delta(E - E_1 - E_2) \\ &\quad = \int d^3\mathbf{p}_2 \frac{1}{\sqrt{k_2^2 + M_2^2}}\delta^3(\mathbf{p}_1 + \mathbf{p}_2)\delta(E - E_1 - \sqrt{k_2^2 + M_2^2}) \\ &\quad = \frac{1}{\sqrt{k_1^2 + M_2^2}}\delta(E - E_1 - \sqrt{k_1^2 + M_2^2}) \end{aligned} \tag{125}$$

Next, we do the integral over the angular components of $\mathbf{p}_1$ by splitting $d^3\mathbf{p}_1 = k_1^2 dk_1 d\Omega_1$, where Ω_1 is the solid angle of $\hat{\mathbf{p}}_1$. The entire integrand now only depends on k_1, except for the spherical harmonics that depend on $\hat{\mathbf{p}}_1$, i.e. Ω_1.

$$\int d\Omega_1 Y_l^m(\Omega_1)Y_{l'}^{m'*}(\Omega_1) = \delta_{l,l'}\delta_{m,m'} \tag{126}$$

Equation 123 now reads

$$\delta^3(\mathbf{p}')\delta(E'-E)\delta_{l,l'}\int dk_1 \frac{Ek_1}{E_1\sqrt{k_1^2+M_2^2}}\delta(E-E_1-\sqrt{k_1^2+M_2^2}) \\ \sum_{\substack{\sigma_1\sigma_2\bar{n},\\ m,\mu,m',\mu'}} \delta_{\bar{n},n}\delta_{\bar{n},n'}\delta_{m,m'}C_{s_1s_2}(s,\mu;\sigma_1\sigma_2)C_{ls}(j,\sigma;m,\mu)C_{s_1s_2}(s',\mu';\sigma_1\sigma_2)C_{l's'}(j',\sigma';m',\mu') \tag{127}$$

The sums can be evaluated using the Clebsch-Gordon identities Weinberg gives in the footnote on page 154. The sum over σ_1, σ_2 gives Kronecker deltas $\delta_{s,s'}\delta_{\mu,\mu'}$. Summing over $m', \mu', \bar{n}$ gives

$$\delta_{s,s'}\delta_{n,n'}\sum_{m,\mu} C_{ls}(j,\sigma;m,\mu)C_{l's'}(j',\sigma';m,\mu) = \delta_{s,s'}\delta_{n,n'}\delta_{j,j'}\delta_{\sigma,\sigma'} \tag{128}$$

For the k_1 integral, one trick to solve this is to convert it into an integral over E_1. From $E_1^2 = k_1^2 + M_1^2$, we have $dE_1 = k_1 dk_1/E_1$, so the integral becomes

$$\int dE_1 \frac{E}{\sqrt{E_1^2-M_1^2+M_2^2}}\delta(E-E_1-\sqrt{E_1^2-M_1^2+M_2^2}) \tag{129}$$

Since the delta function is 0 when $E = E_1 + \sqrt{E_1^2-M_1^2+M_2^2}$, we can rewrite the integral as

$$\int dE_1 \frac{E_1+\sqrt{E_1^2-M_1^2+M_2^2}}{\sqrt{E_1^2-M_1^2+M_2^2}}\delta(E-E_1-\sqrt{E_1^2-M_1^2+M_2^2}) \tag{130}$$

Let's remind ourselves of the identiy for delta function composition,

$$\delta(g(x)) = \sum_{x_0}\frac{\delta(x-x_0)}{|g'(x_0)|} \tag{131}$$

where x_0 are the zeroes of $g(x)$.

The only value of E_1 that makes the argument of the delta function zero is

$$E_0 = \frac{E^2+M_1^2-M_2^2}{2E} \tag{132}$$

Therefore $|g'(E_0)|$ is

$$|g(E_0)| = \frac{E_0+\sqrt{E_0^2-M_1^2+M_2^2}}{\sqrt{E_0^2-M_1^2+M_2^2}} \tag{133}$$

Finally, the full E_1 integral evaluates to 1:

$$\int dE_1\delta(E_1-E_0)\frac{E_1+\sqrt{E_1^2-M_1^2+M_2^2}}{\sqrt{E_1^2-M_1^2+M_2^2}}\frac{\sqrt{E_0^2-M_1^2+M_2^2}}{E_0+\sqrt{E_0^2-M_1^2+M_2^2}} = 1 \tag{134}$$

Equation 127 is therefore reduced to the right hand side of Equation 122, so have we proved

$$(\Phi_{E'\mathbf{p}'j'\sigma'l's'n'}, \Phi_{E0j\sigma lsn}) = \delta^3(\mathbf{p}')\delta(E'-E)\delta_{j',j}\delta_{\sigma',\sigma}\delta_{l',l}\delta_{s',s}\delta_{n',n} \tag{135}$$

4 Chapter 4

4.1 Problem 4.1 – Generating Functionals

Important equations for the question

$$\begin{aligned} F[v] \equiv 1 + \sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int v^*(q_1')...v^*(q_N')v(q_1)...v(q_M) \\ \times S_{q_1'...q_N',q_1...q_M}dq_1'...dq_N'dq_1...dq_M \end{aligned} \tag{136}$$

$$\begin{aligned} F^C[v] \equiv \sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int v^*(q_1')...v^*(q_N')v(q_1)...v(q_M) \\ \times S^C_{q_1'...q_N',q_1...q_M}dq_1'...dq_N'dq_1...dq_M \end{aligned} \tag{137}$$

$$S_{\beta\alpha} = \sum_{PART}(\pm)S^C_{\beta_1\alpha_1}S^C_{\beta_2\alpha_2}... \tag{138}$$

Because we only need to consider the bosonic case, we can discard the $\pm$ from now on. In order to make the notation nicer, I'm going to introduce the following notation:

$$\alpha = q_1...q_M \tag{139}$$

$$\beta = q_1'..q_N' \tag{140}$$

$$v^*(q_1')...v^*(q_N') = \prod_{\beta}^{N} v^*(q_\beta') \tag{141}$$

$$v(q_1)...v(q_M) = \prod_{\alpha}^{N} v(q_\alpha) \tag{142}$$

$$C_{NM} = \int \prod_{\beta}^{N} v^*(q_\beta)\prod_{\alpha}^{M} v(q_\alpha)S^C_{\alpha\beta}d\beta d\alpha \tag{143}$$

$$F^C[v] = \sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{C_{NM}}{N!M!} \tag{144}$$

$$F[v] = 1 + \sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int \prod_{\beta}^{N} v^*(q_\beta')\prod_{\beta}^{N} v^*(q_\beta')S_{\beta\alpha}d\beta d\alpha \tag{145}$$

We can now replace the S matrix in the expression for $F[v]$ with the definition of the connected S matrices

$$\begin{aligned} F[v] = 1 + \sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\int \prod_{\beta}^{N} v^*(q_\beta')\prod_{\alpha}^{M} v(q_\alpha) \\ \times\left(\sum_{PART}S^C_{\beta_1\alpha_1}S^C_{\beta_2\alpha_2}...\right)d\beta d\alpha \end{aligned} \tag{146}$$

factorizing the equation yields:

$$\begin{aligned} F[v] = 1 + \sum_{N=1}^{\infty}\sum_{M=1}^{\infty}\frac{1}{N!M!}\sum_{PART} \\ \times\left(\int \prod_{\beta_1}^{|\beta_1|} v^*(q_{\beta_1}')\prod_{\alpha_1}^{|\alpha_1|} v(q_{\alpha_1})S^C_{\beta_1\alpha_1}d\alpha_1 d\beta_1\right) \\ \times\left(\int \prod_{\beta_2}^{|\beta_2|} v^*(q_{\beta_2}')\prod_{\alpha_2}^{|\alpha_2|} v(q_{\alpha_2})S^C_{\beta_2\alpha_2}d\alpha_2 d\beta_2\right)... \end{aligned} \tag{147}$$

This can be written much more compressed using the notation introduced earlier because the qs are just integration variables:

$$F[v] = 1 + \sum_{N=1}^{\infty}\sum_{M=1}^{\infty} \frac{1}{N!M!} \sum_{PART} C_{|\beta_1||\alpha_1|} C_{|\beta_2||\alpha_2|} \cdots \tag{148}$$

Keep in mind

$$\sum_i |\beta_i| = N$$
$$\sum_i |\alpha_i| = M$$

From Eq. 148, we can pull out the first term of the sum over partitions to get an expression for $F[v]$ in terms of $F^C[v]$

$$F[v] = 1 + \sum_{N=1}^{\infty}\sum_{M=1}^{\infty} \frac{1}{N!M!} \left(C_{NM} + \sum_{PART}{}' C_{|\beta_1||\alpha_1|} C_{|\beta_2||\alpha_2|} \cdots \right) \tag{149}$$

$$F[v] = 1 + F^C[v] + \sum_{N=1}^{\infty}\sum_{M=1}^{\infty} \frac{1}{N!M!} \sum_{PART}{}' C_{|\beta_1||\alpha_1|} C_{|\beta_2||\alpha_2|} \cdots \tag{150}$$

Where $\sum'$ is the sum over products of two or more C.

To get a more general relationship, let us return to the expression including the sum over all partitions:

$$F[v] = 1 + \sum_{N=1}^{\infty}\sum_{M=1}^{\infty} \frac{1}{N!M!} \sum_{PART} C_{|\beta_1||\alpha_1|} C_{|\beta_2||\alpha_2|} \cdots \tag{151}$$

Because the summand does not depend on the the specific partitions, just the lengths of the partitions, I can sum over sets of multiplicities of permutations $\{m_{a,b}\}$, where a corresponds to the number of outgoing particles and b corresponds to the number of incoming particles. Thus, when I sum over the number of incoming and outgoing particles

$$\sum_{a,b} a \cdot m_{a,b} = N$$
$$\sum_{a,b} b \cdot m_{a,b} = M$$

Thus, I can convert the double sum over all total number of outgoing and incoming particles to a sum over all sets of multiplicities of particles grouped together into all permutations. Now the question is the combinatorial factor for how many ways the particles can be grouped together. There are $N!M!$ different ways of listing the particles, this determines which particles get sorted into what connected piece. Inside each connected piece, the order does not matter, so you must divide by $a!b!$ to remove duplicates. For a certain multiplicity of that cluster appearing, you must divide by this factor each time, so you get a factor of $(a!b!)^m_{a,b}$. Also, each cluster is indistinguishable from each other cluster with the same length, so you must divide by a factor of $m_{a,b}!$. Each set $\{m_{a,b}\}$ is composed of multiple groupings where these factors must be divided out, so you must divide by the product of $m_{a,b}!(a!b!)^{m_{a,b}}$ over all a and b. Thus the combinatorial factor becomes:

$$\frac{N!M!}{\prod_{a,b} m_{a,b}(a!b!)^{m_{a,b}}}$$

Now, the sums over total number of incoming and outgoing particles can be replaced by a single sum over sets of multiplicities of groupings of $|\beta_i|$ outgoing particles and $|\alpha_i|$ incoming particles.

$$F[v] = 1 + \frac{1}{N!M!} \sum_{\{m_{|\beta_i|,|\alpha_i|}\}} \frac{M!N!}{\prod_{|\beta_i|,|\alpha_i|} m_{|\beta_i|,|\alpha_i|}!(|\beta_i|!|\alpha_i|!)^{m_{|\beta_i|,|\alpha_i|}}} \prod_{|\beta_i|,|\alpha_i|} C_{|\beta_i||\alpha_i|}^{m_{|\beta_i|,|\alpha_i|}} \tag{152}$$

Simplifying

$$F[v] = \prod_{|\beta_i|,|\alpha_i|} \sum_{m_{|\beta_i|,|\alpha_i|}=0} \frac{1}{m_{|\beta_i|,|\alpha_i|}!(|\beta_i|!|\alpha_i|!)^{m_{|\beta_i|,|\alpha_i|}}} C_{|\beta_i||\alpha_i|}^{m_{|\beta_i|,|\alpha_i|}} \tag{153}$$

$$F[v] = \prod_{|\beta_i|,|\alpha_i|} e^{C_{|\beta_i||\alpha_i|}/|\beta_i|!|\alpha_i|!} \tag{154}$$

$$F[v] = \exp\left(\sum_{\beta_i=0}^{\infty} \sum_{\alpha_i=0}^{\infty} \frac{C_{|\beta_i||\alpha_i|}}{|\beta_i|!|\alpha_i|!}\right) \tag{155}$$

Finally,

$$F[v] = e^{F^C[v]} \tag{156}$$

4.2 Problem 4.2 – Spin-0 Particle Interactions

We wish to calculate the S-matrix element $S_{\beta\alpha}$ and differential cross section for scattering of spinless bosons with mass $M > 0$ in the center-of-mass frame to order g given an interaction:

$$V = g \int d^3\mathbf{p}_1\, d^3\mathbf{p}_2\, d^3\mathbf{p}_3\, d^3\mathbf{p}_4\, \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) a(\mathbf{p}_3) a(\mathbf{p}_4) \tag{157}$$

We start with the definition of the S-matrix in time-dependent pertubation theory to first order in V:

$$S = 1 - i\int_{-\infty}^{\infty} dt\, V(t) = 1 - i\int_{-\infty}^{\infty} dt\, e^{iH_0 t} V e^{-iH_0 t} \tag{158}$$

Taking the inner product with states Φ_β and Φ_α, we have:

$$S_{\beta\alpha} = \delta(\beta - \alpha) - i\int_{-\infty}^{\infty} dt\, e^{i(E_\beta - E_\alpha)t} (\Phi_\beta, V\Phi_\alpha) \tag{159}$$

We now wish to calculate

$$(\Phi_\beta, V\Phi_\alpha) = g \int d^3\mathbf{p}_1\, d^3\mathbf{p}_2\, d^3\mathbf{p}_3\, d^3\mathbf{p}_4\, \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)(\Phi_\beta, a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) a(\mathbf{p}_3) a(\mathbf{p}_4)\Phi_\alpha) \tag{160}$$

which lets us narrow our focus to just calculating

$$(\Phi_\beta, a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) a(\mathbf{p}_3) a(\mathbf{p}_4)\Phi_\alpha) = (a(\mathbf{p}_2) a(\mathbf{p}_1)\Phi_\beta, a(\mathbf{p}_3) a(\mathbf{p}_4)\Phi_\alpha) \tag{161}$$

Let's now formulate conditions on when $a(\mathbf{p}_3)a(\mathbf{p}_4)\Phi_\alpha = 0$ and what happens when both the bra and ket are non-zero in the inner product. In general, we can describe our state Φ_α, which let's say consists of N_α bosons, as

$$\Phi_\alpha = a^\dagger(\mathbf{k}_1) \ldots a^\dagger(\mathbf{k}_{N_\alpha})\Phi_0 \tag{162}$$

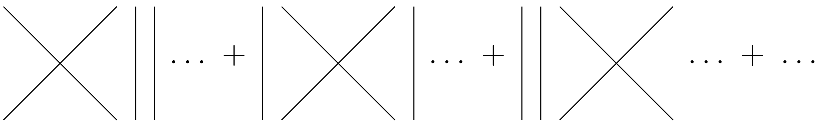
For generality, let's suppose we have a product of M annihilation operators acting on this state. This would look like

$$a(\mathbf{p}_1) \ldots a(\mathbf{p}_M) a^\dagger(\mathbf{k}_1) \ldots a^\dagger(\mathbf{k}_{N_\alpha})\Phi_0 \tag{163}$$

If we pass one annihilation operator through all N_α creation operators, we are left with a sum where each term is a product of a single delta function and $N_\alpha - 1$ creation operators acting on the vacuum Φ_0. That is, unless $N_\alpha = 0$ where we would just get 0. We can repeat this process for M annihilation operators, where we would get a sum of terms where each has M deltas and $N_\alpha - M$ creation operators acting on Φ_0, unless $M > N_\alpha$, where we would just get 0. Since we are operating on Φ_α with $M = 2$ annihilation operators, the number of particles in Φ_α must be ≥ 2 in order to get a non-zero contribution to first-order in pertubation theory. Similarly, we find $N_\beta \geq 2$.

Now, after we pass all the annihilation operators through Equation 163, and assuming $N_\alpha \geq M$, each of the resulting terms can be expressed as M delta functions acting on a state with $N_\alpha - M$ particles. This is due to reapplying the creation operators on the vacuum. This implies that when taking inner products of two such states given by Equation 163, we require $N_\beta = N_\alpha \equiv N$.

Finally, let's investigate how this interaction look diagramatically. We will have one vertex because we are at first order in pertubation theory. Our interaction has two annihilation operators, so we get two lines going up and meeting at the vertex when we move these operators through the Φ_α creation operators. We also have two creation operators in V, so we get two lines exiting the vertex when the Φ_β adjoint annihilation operators move past the V creation operators. We are left with $N-2$ annihilation operators from Φ_β adjoint and $N-2$ creation operators from Φ_α, so we get $N-2$ lines going straight up through the diagram as we pass the annihilation operators through the creation operators. This looks like something like:

$(\Phi_\beta, V\Phi_\alpha) =$  (164)

where the vertical lines occur $N-2$ times. This is equivalent to the Equation (4.4.5) in Weinberg to first order (minus the time dependence):

$$(\Phi_\beta, V\Phi_\alpha) = \sum_{\text{clusterings}} \prod_j (\Phi_{\beta_j}, V_j \Phi_{\alpha_j})_{\mathrm{C}} \tag{165}$$

with $V_j = 1$ for clusters that do not involve a vertex. For clusters that do not involve a vertex, we are simply left with $\delta(\beta_j - \alpha_j)$, so we only need to concern ourselves with the diagram that contains a vertex, i.e. the one that corresponds to $N_\alpha = N_\beta = 2$ scattering. Returning to Equation 161, we want to calculate:

$$(a(\mathbf{p}_2)a(\mathbf{p}_1)\Phi_\beta, a(\mathbf{p}_3)a(\mathbf{p}_4)\Phi_\alpha) = (a(\mathbf{p}_2)a(\mathbf{p}_1)a^\dagger(\mathbf{k}_1')a^\dagger(\mathbf{k}_2')\Phi_0, a(\mathbf{p}_3)a(\mathbf{p}_4)a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\Phi_0) \tag{166}$$

Focusing on one vector in the inner product:

$$\begin{aligned} a(\mathbf{p}_3)a(\mathbf{p}_4)a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\Phi_0 &= a(\mathbf{p}_3)a^\dagger(\mathbf{k}_1)a(\mathbf{p}_4)a^\dagger(\mathbf{k}_2)\Phi_0 + \delta^3(\mathbf{p}_4 - \mathbf{k}_1)a(\mathbf{p}_3)a^\dagger(\mathbf{k}_2)\Phi_0 \\ &= a(\mathbf{p}_3)a^\dagger(\mathbf{k}_1)a(\mathbf{p}_4)a^\dagger(\mathbf{k}_2)\Phi_0 + \delta^3(\mathbf{p}_4 - \mathbf{k}_1)\delta^3(\mathbf{p}_3 - \mathbf{k}_2)\Phi_0 \\ &= \delta^3(\mathbf{p}_4 - \mathbf{k}_2)\delta^3(\mathbf{p}_3 - \mathbf{k}_1)\Phi_0 + \delta^3(\mathbf{p}_4 - \mathbf{k}_1)\delta^3(\mathbf{p}_3 - \mathbf{k}_2)\Phi_0 \\ &= \left[\delta^3(\mathbf{p}_3 - \mathbf{k}_1)\delta^3(\mathbf{p}_4 - \mathbf{k}_2) + \delta^3(\mathbf{p}_3 - \mathbf{k}_2)\delta^3(\mathbf{p}_4 - \mathbf{k}_1)\right]\Phi_0 \end{aligned} \tag{167}$$

Similarly, we have

$$a(\mathbf{p}_2)a(\mathbf{p}_1)a^\dagger(\mathbf{k}_1')a^\dagger(\mathbf{k}_2')\Phi_0 = \left[\delta^3(\mathbf{p}_1 - \mathbf{k}_1')\delta^3(\mathbf{p}_2 - \mathbf{k}_2') + \delta^3(\mathbf{p}_1 - \mathbf{k}_2')\delta^3(\mathbf{p}_2 - \mathbf{k}_1')\right]\Phi_0 \tag{168}$$

Equation 166 reduces to:

$$\left[\delta^3(\mathbf{p}_1 - \mathbf{k}_1')\delta^3(\mathbf{p}_2 - \mathbf{k}_2') + \delta^3(\mathbf{p}_1 - \mathbf{k}_2')\delta^3(\mathbf{p}_2 - \mathbf{k}_1')\right]\left[\delta^3(\mathbf{p}_3 - \mathbf{k}_1)\delta^3(\mathbf{p}_4 - \mathbf{k}_2) + \delta^3(\mathbf{p}_3 - \mathbf{k}_2)\delta^3(\mathbf{p}_4 - \mathbf{k}_1)\right] \tag{169}$$

Therefore, the matrix element of the time-independent interaction potential for $N = 2$ is

$$(\Phi_\beta, V\Phi_\alpha) = 4g\,\delta^3(\mathbf{k}_1' + \mathbf{k}_2' - \mathbf{k}_1 - \mathbf{k}_2) \tag{170}$$

The connected S-matrix element for $N = 2$ is

$$\begin{aligned} S_{\beta\alpha} &= \delta(\beta-\alpha) - 4ig\,\delta^3(\mathbf{k}_1' + \mathbf{k}_2' - \mathbf{k}_1 - \mathbf{k}_2)\int_{-\infty}^{\infty} dt\, e^{i(E_1'+E_2'-E_1-E_2)t} \\ S_{\beta\alpha} &= \delta(\beta-\alpha)(1-8i\pi g) \\ &\boxed{\implies S^{\mathrm{C}}_{\beta\alpha} = -2i\pi\delta(\beta-\alpha)(4g)} \end{aligned} \tag{171}$$

where we used Weinberg's result for two particle states $S^{\mathrm{C}}_{\beta\alpha} = (S-1)_{\beta\alpha}$.

The full S-matrix element is then:

$$S_{\beta\alpha} = \delta(\beta-\alpha)\left[1 - 8i\pi g \sum_{\text{clusterings}}\right] \tag{172}$$

The sum over clusters is simply the number of ways of putting N particles into unordered groups with one group of two particles and $N-2$ groups of one particle each. This is the same as the number of ways of putting N particles into an unordered group of two, which is just N choose 2, so

$$\boxed{S_{\beta\alpha} = \delta(\beta-\alpha)\left[1 - 4i\pi g N(N-1)\right]} \tag{173}$$

When calculating the cross section, we must restrict ourselves to only states Φ_α and Φ_β that have no subset of particles whose momenta are unchanged during the interaction. Therefore, we should only consider the connected S-matrix consisting of two particles. The delta function-free matrix element is

$$M_{\beta\alpha} = 4g \tag{174}$$

which, using Weinberg Equation (3.4.30), leads to the cross section

$$\begin{aligned} \frac{d\sigma(\alpha\to\beta)}{d\Omega} &= \frac{(2\pi)^4 k' E_1' E_2' E_1 E_2}{E^2 k}|4g|^2 \\ &\boxed{\frac{d\sigma(\alpha\to\beta)}{d\Omega} = g^2\frac{(4\pi)^4 k' E_1' E_2' E_1 E_2}{E^2 k}} \end{aligned} \tag{175}$$