Weinberg Volume 1 Chapter Summaries

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Contents

1 Introduction

This is a companion document to the Weinberg solutions set that myself and a few other Berkeley students are writing. My hope with writing chapter summaries is to help my understanding as I read, but also to provide a concise reference to how Weinberg arrives at certain conclusions. Special attention will be paid to the logical structure of *The Quantum Theory of Fields*, as that is what I think sets this textbook apart from the many others.

I will follow all the same conventions as Weinberg does. Most importantly, a generic four vector is written as $A^\mu = (A^1, A^2, A^3, A^0)$, and the metric $\eta^{\mu\nu} = \text{diag}(1, 1, 1, -1)$. I will also use natural units, unless explicitly stated otherwise.

2 Chapter 2 - Relativistic Quantum Mechanics

2.1 Quantum Mechanics and Special Relativity

Quantum field theory takes as axiomatic two theories of the early 1900s: quantum mechanics and special relativity. This is a good assumption, since we have great experimental evidence in support of these theories. The postulates of quantum mechanics, as presented by Weinberg, are:

(i) Physical states are represented by a ray $\mathcal{R}$ in a complex vector space $\mathcal{E}$ called a Hilbert space. $\mathcal{R}$ is the set of all normalized vectors Ψ that differ only by a phase.

(ii) Observables are represented by Hermitian operators $A = A^\dagger$ on $\mathcal{E}$.

(iii) The probability of finding a state $\mathcal{R}$ in a (potentially) new state $\mathcal{R}'$ is

$$P(\mathcal{R} \to \mathcal{R}') = |(\Psi, \Psi')|^2 \tag{1}$$

where $\Psi \in \mathcal{R}$ and $\Psi' \in \mathcal{R}'$.

The classic statement of the postulates of special relativity is:

(i) Physical laws are the same in all inertial reference frames.

(ii) The speed of light in vacuum is 1 in all inertial reference frames.

These two postulates can be casted mathematically into the statement:

(i) The equations of the laws of physics are Poincare covariant. That is, any physical law can be built out of scalars, vectors, tensors, etc. that transform under representations of the Poincare group.

The importance of special relativity is that its postulates impose a symmetry on our Hilbert space which constrains the form our states and operators take. *The Quantum Theory of Fields* really just follows the logical conclusion of what happens when we limit the form quantum mechanics can take.

2.2 The Poincare Group, its Algebra, and its Representations

From Wigner's theorem, *any* symmetry can be represented by a linear, unitary operator or antilinear, antiunitary operator. Symmetries dependent on a continuous parameter must be represented only by a linear, unitary operator, since they can be continuously changed into the identity operator which is linear and unitary.

Symmetry transformations form a group, but since the transformations map rays into other rays, our group multiplication law for the representations has the form:

$$U(T_2)U(T_1) = e^{i\phi(T_2,T_1)}U(T_2T_1) \tag{2}$$

for $T_1, T_2 \in G$, where G is some symmetry group. This inclusion of a phase has very important consequences that we will see later in terms of the allowed angular momenta of particles. Importantly, it is a consequence of the postulates of quantum mechanics dictating our states are represented by rays.

Connected Lie groups are groups defined by real, continuous parameters with all elements connected to the identity by a path within the group. The group multiplication law takes the form:

$$U(T(\bar\theta))U(T(\theta)) = \exp\Big(i\phi(T(\bar\theta), T(\theta))\Big)U(T(f(\bar\theta, \theta))) \tag{3}$$

When we go to infinitessimal parameters $\theta, \bar\theta$, we find the following relation:

$$[t_b, t_c] = iC^a{}_{bc}t_a + iC_{bc}1 \tag{4}$$

where t_a are our generators, $C^a{}_{bc}$ are our structure constants defined as

$$f^a(\bar\theta, \theta) = \theta^a + \bar\theta^a + f^a{}_{bc}\bar\theta^b\theta^c + ..., \quad C^a{}_{bc} \equiv -f^a{}_{bc} + f^a{}_{cb}, \tag{5}$$

and C_{bc} are our central charges defined as

$$\phi(T(\bar\theta), T(\theta)) = f_{ab}\bar\theta^a\theta^b + ..., \quad C_{bc} = -f_{bc} + f_{cb} \tag{6}$$

The commutation relations given by Equation 4 are called a Lie algebra. One could ask the question: are the phases ϕ in our representations fundamental to our group or could they be set to 0 if we were to smartly choose a different representation? A special theorem exists that says we can choose $\phi = 0$ if both of the following are met:

(a) The generators of the group can be redefined as to eliminate the central charges from the Lie algebra. This only happens if we can write $C_{ab} = C^{e}{}_{ab}\phi_e$ for real constants ϕ_e. The redefinition is then given by $\tilde{t_a} \equiv t_a + \phi_a$.

(b) The group is simply connected, so that any closed path within the group can be continuously deformed into a point.

At this stage, we now want to analyze the Lie algebra, central charges, and topology of the Poincare group. Starting with the Poincare algebra, we find (assuming no central charges exist):

$$\begin{aligned} i[J^{\mu\nu}, J^{\rho\sigma}] &= \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu} \\ i[P^{\mu}, J^{\rho\sigma}] &= \eta^{\mu\rho}P^{\sigma} - \eta^{\mu\sigma}P^{\rho} \\ [P^{\mu}, P^{\rho}] &= 0 \end{aligned} \tag{7}$$

Weinberg shows in Chapter 2.7 that when we add central charges into the Poincare algebra, they can be eliminated by redefining the generators. However, the Galilean algebra does admit a central charge, namely the mass M. This is shown in the solutions to Problem 2.3.

The topology of the Poincare group is not simply connected, however. Weinberg shows the topology of the Poincare group is $R_4 \times R_3 \times S_3/Z_2$. The fundamental group is Z_2, corresponding to the two classes of loops through the space. The first class is loops that that go an odd number of times over the same path from an element back to itself; the second class goes an even number of times. This is because a double loop over the same path can be continually contracted to a point. Therefore, we have the identity:

$$[U(\Lambda)U(\bar{\Lambda})U^{-1}(\Lambda\bar{\Lambda})]^2 = 1 \tag{8}$$

Therefore, the projective phase for the Poincare group is

$$e^{i\phi} = \pm 1 \tag{9}$$

We will return to this phase when we talk about particles.

2.3 Particles

3 Chapter 3 - Scattering Theory