Solutions to Steven Weinberg's *the Quantum Theory of Fields* Volume 1

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Chapter 2

Problem 2.1

We will make use of equation (2.5.23) for how a massive particle state $\Psi_{p,\sigma}$ transforms under a homogenous Lorentz transformation $U(\Lambda)$:

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(W(\Lambda, p))\Psi_{\Lambda p,\sigma'} \tag{1}$$

The most difficult part of this problem is finding what the little group transformation W is. W is given by equation (2.5.10):

$$W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p) \tag{2}$$

Before we compute W, we can note two properties it must have. Since the little group for massive particles is $SO(3)$, we know that W, a representation of the little group, must be a rotation matrix. The other property is the rotation matrix must be a rotation about the x-axis. This is because $\vec{p}$ is in the y-direction, and therefore the boost $L(p)$ preserves four-vectors' x-components. Similarly, the boost Λ is in the z-direction and preserves x-components. The boost $L^{-1}(\Lambda p)$ boosts in the y- and z-directions, and must also preserve x-components. So the rotation W must leave x-components invariant, which means the rotation must be about the x-axis.

The energy of the W-boson in observer $\mathcal{O}$'s frame is $E = \sqrt{p^2 + m^2}$, and therefore the four-momentum is

$$p^\mu = (0, p, 0, E) \tag{3}$$

We will use equation (2.5.24) to calcualte $L(p)$ and $L^{-1}(\Lambda p)$. The Lorentz factor to go from k^μ to p^μ is $\gamma = \frac{E}{m}$, so $\sqrt{\gamma^2 - 1} = \frac{p}{m}$. The unit three-momenta are $\hat{p_1} = \hat{p_3} = 0$ and $\hat{p_2} = 1$. The boost $L(p)$ is then

$$L(p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{E}{m} & 0 & \frac{p}{m} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{p}{m} & 0 & \frac{E}{m} \end{bmatrix} \tag{4}$$

Since $\mathcal{O}'$ is moving at speed v in the $+z$-direction relative to $\mathcal{O}$, the boost that takes us from $\mathcal{O}$ to $\mathcal{O}'$ is

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -v\gamma \\ 0 & 0 & -v\gamma & \gamma \end{bmatrix} \tag{5}$$

where $\gamma = \frac{1}{\sqrt{1-v^2}}$. Note that this γ is not referring to the gamma used to previously find $L(p)$, but rather refers to the boost from $\mathcal{O}$ to $\mathcal{O}'$.

The four-momenta to $\mathcal{O}'$ is

$$(\Lambda p)^\mu = (0, p, -v\gamma E, \gamma E) \tag{6}$$

The boost $L^{-1}(\Lambda p)$ is the inverse of $L(\Lambda p)$, and therefore boosts a particle with four-momentum $(\Lambda p)^\mu$ back into its rest frame. This is equivalent to boosting the particle in the opposite direction it was originally boosted in, so $L^{-1}(\Lambda p) = L(-\Lambda p)$. The Lorentz factor for this boost is $\gamma = \frac{E'}{m} = \frac{\gamma E}{m}$. The expression for L^i_0 can be simplified when solving for these components:

$$L^i_0(p) = \hat{p_i}\sqrt{\gamma^2 - 1} = \frac{p_i}{|\vec{p}|}\frac{|\vec{p}|}{m} = \frac{p_i}{m} \tag{7}$$

Therefore

$$L^1_0(-\Lambda p) = 0 \tag{8}$$

$$L^2_0(-\Lambda p) = -\frac{p}{m} \tag{9}$$

$$L^3_0(-\Lambda p) = \frac{v\gamma E}{m} \tag{10}$$

The boost $L^{-1}(\Lambda p)$ then reads

$$L^{-1}(\Lambda p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\gamma E}{m}\left(\frac{p^2+v^2\gamma m E}{p^2+v^2\gamma^2E^2}\right) & \frac{v\gamma p E}{m}\left(\frac{m-\gamma E}{p^2+v^2\gamma^2E^2}\right) & -\frac{p}{m} \\ 0 & \frac{v\gamma p E}{m}\left(\frac{m-\gamma E}{p^2+v^2\gamma^2E^2}\right) & \frac{v^2\gamma^3E^3+mp^2}{m(p^2+v^2\gamma^2E^2)} & \frac{v\gamma E}{m} \\ 0 & -\frac{p}{m} & \frac{v\gamma E}{m} & \frac{\gamma E}{m} \end{bmatrix} \tag{11}$$

Plugging all of this into Equation 2 gives the full little group element

$$W(\Lambda, p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\gamma m+E}{m+\gamma E} & \frac{v\gamma p}{m+\gamma E} & 0 \\ 0 & -\frac{v\gamma p}{m+\gamma E} & \frac{\gamma m+E}{m+\gamma E} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

We should note that $W(\Lambda, p)$ has the predicted form of a rotation matrix about the x-axis, where we identify $\cos(\theta) = \frac{\gamma m+e}{m+\gamma E}$ and $\sin(\theta) = \frac{v\gamma p}{m+\gamma E}$.

Since the W-boson is a spin-1 particle, the representation $D^{(j=1)}_{\sigma'\sigma}$ of $W(\Lambda, p)$ is simply a rotation matrix for 3D vectors, so we can immediately identify

$$D^{(1)}_{\sigma'\sigma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\gamma m+E}{m+\gamma E} & \frac{v\gamma p}{m+\gamma E} \\ 0 & -\frac{v\gamma p}{m+\gamma E} & \frac{\gamma m+E}{m+\gamma E} \end{bmatrix} \tag{13}$$

with subsequent rows and columns numbered -1, 0, and 1.

We are now able to write the full transformed state

$$U(\Lambda)\Psi_{p,+1} = \sqrt{\frac{\gamma E}{E}} \sum_{\sigma'} D^{(1)}_{\sigma',+1}(W(\Lambda,p))\Psi_{\Lambda p,\sigma'} \tag{14}$$

$$= \sqrt{\gamma}\left(D^{(1)}_{-1,+1}\Psi_{\Lambda p,-1} + D^{(1)}_{0,+1}\Psi_{\Lambda p,0} + D^{(1)}_{+1,+1}\Psi_{\Lambda p,+1}\right) \tag{15}$$

$$= \sqrt{\gamma}\left(\frac{v\gamma p}{m+\gamma E}\Psi_{\Lambda p,0} + \frac{\gamma m + E}{m+\gamma E}\Psi_{\Lambda p,+1}\right) \tag{16}$$

$$\boxed{U(\Lambda)\Psi_{p,+1} = \frac{\sqrt{\gamma}}{m+\gamma E}\left(v\gamma p\Psi_{\Lambda p,0} + (\gamma m + E)\Psi_{\Lambda p,+1}\right)} \tag{17}$$

We can further check that when $v \to 0$, we get that $U(\Lambda)\Psi_{p,+1} = \Psi_{p,+1}$, as expected.

Problem 2.3

This has been covered in Hagimoto already.

Problem 2.3